

Solution to Final Examination

MAT 1300C, Winter 2011

Section A: Multiple Choice Questions (2 × 10 = 20 marks)

1. The equation of the tangent line of the function $y = xe^x$ at the point where $x = 1$ is

- (A) $y = ex + 1$; (B) $y = ex - 1$; (C) $y = 2ex + 1$;
 (D) $y = 2ex - 1$; (E) $y = 2ex + e$; (F) $y = 2ex - e$.

Solution. (F) The derivative of y is $y' = e^x + xe^x$. When $x = 1$, $y' = 2e$. The equation is of the form $y = 2ex + b$. Since $y(1) = e = 2e + b$, $b = -e$. The equation of the tangent line is $y = 2ex - e$.

2. The derivative of a function $y = f(x)$ defined implicitly by the equation $x^2y^2 + 3x - 2y = 3$ at the point $x = 1$ and $y = 2$ is

- (A) 2; (B) -2; (C) 7 / 2; (D) -7 / 2; (E) 11 / 2; (F) -11 / 2.

Solution. (F) Take the derivative with respect to x on both sides with the chain rule:

$2xy^2 + 2x^2yy' + 3 - 2y' = 0$. When $x = 1$, and $y = 2$, $8 + 4y' + 3 - 2y' = 0$, $2y' + 11 = 0$, $y' = -11/2$.

3. Suppose the revenue function of a product is $R = 30x - \frac{8}{\sqrt{x}}$, and the cost function of this product is $C = 10x + 100$. Then the marginal profit at $x = 4$ is

- (A) 20.5; (B) 19.5; (C) 7.5; (D) 13.5; (E) 6.5; (F) 10.5.

Solution. (A) The profit function is $P = 30x - \frac{8}{\sqrt{x}} - 10x - 100 = 20x - \frac{8}{\sqrt{x}} - 100$. The marginal profit is $P' = 20 + 4x^{-3/2}$. When $x = 4$, $P'(4) = 20 + 1/2 = 20.5$.

4. Consider the function $f(x) = x^3 - 3x^2 - 9x$. On which interval(s) is this function increasing?

- (A) $x < 1$ and $x > 3$ only; (B) $-1 < x < 3$ only; (C) $x > -3$ only;
 (D) $x > 1$ only; (E) $x < -1$ and $x > 3$ only; (F) $1 < x < 3$ only.

Solution. (E). Let $y' = 3x^2 - 6x - 9 = 0$. $x^2 - 2x - 3 = 0$. $x = -1, x = 3$. Since $y' > 0$ on $(-\infty, -1)$, and $(3, \infty)$, this function is increasing on these intervals.

5. Consider the function $f(x) = 3x^4 - 10x^3 - 12x^2 + 2$. On which interval(s) is this function concave down?

- (A) $x < -1/3$ only; (B) $x < -1/3$ or $x > 2$ only; (C) $x > 2$ only;
 (D) $-1/3 < x < 2$ only; (E) $x < 2$ only; (F) $x > -1/3$ only.

Solution. (D). $f'(x) = 12x^3 - 30x^2 - 24x$. $f''(x) = 36x^2 - 60x - 24$. Let $f''(x) = 0$. $3x^2 - 5x - 2 = 0$. $x = 2$, $x = -1/3$. $f''(x) < 0$ when x in $(-1/3, 2)$.

6. Consider function $y = \frac{x^3}{3} - x^2 - 3x$ defined on the interval $-2 \leq x \leq 6$. Which one of the following statements is true?

- (A) The absolute maximum value of y is 18, and the absolute minimum value of y is -9 .
 (B) The absolute maximum value of y is 18, and the absolute minimum value of y is $-\frac{2}{3}$.
 (C) The absolute maximum value of y is $\frac{5}{3}$, and the absolute minimum value of y is -9 .
 (D) The absolute maximum value of y is 36, and the absolute minimum value of y is $-\frac{2}{3}$.
 (E) The absolute maximum value of y is $\frac{5}{3}$, and the absolute minimum value of y is -18 .
 (F) The absolute maximum value of y is 36, and the absolute minimum value of y is -18 .

Solution. (A) $y' = x^2 - 2x - 3$. Let $y' = 0$. $x = -1, 3$. $y' > 0$ when $-2 < x < -1$ or $3 < x < 6$; $y' < 0$ when $-1 < x < 3$. y attains a relative maximum at $x = -1$. $y(-1) = -\frac{1}{3} - 1 + 3 = \frac{5}{3}$. y attains a relative minimum at $x = 3$. $y(3) = 9 - 9 - 9 = -9$. $y(-2) = -\frac{8}{3} - 4 + 6 = -\frac{2}{3}$, $y(6) = 72 - 36 - 18 = 18$. The absolute maximum is $y(6) = 18$, and the absolute minimum is $y(3) = -9$.

7. The population of a country in 2000 is 1 million, and its population is increased to 1.1 million in 2010. Suppose the population grows exponentially, which of the following is used to calculate the population of the country in 2030?

- (A) $1.1 \times e^3$; (B) $1.1 \times e^{-3}$; (C) $e^{-1.1}$;
 (D) 1.1^3 ; (E) 1.1^{30} ; (F) $e^{1.1}$.

Solution. (D) $P(t) = e^{kt}$. $P(10) = 1.1 = e^{10k}$. $e^{10k} = 1.1$. The population in 2030 is $P(30) = e^{30k} = 1.1^3$.

8. Assume the demand function of a product is $p = 90 - \sqrt{x}$, $0 \leq x \leq 8100$. On which interval is the demand elastic?

- (A) $0 \leq x < 5000$; (B) $5000 < x \leq 8100$; (C) $0 \leq x < 3600$;
 (D) $2500 < x \leq 8100$; (E) $0 \leq x < 2500$; (F) $3600 < x \leq 8100$.

Solution. (C) The elasticity of the demand is

$$\eta = \frac{p/x}{dp/dx} = \frac{(90-\sqrt{x})/x}{-1/(2\sqrt{x})} = -\frac{2(90-\sqrt{x})}{\sqrt{x}}, |\eta| = \frac{2(90-\sqrt{x})}{\sqrt{x}} > 1, 180 - 2\sqrt{x} > \sqrt{x}, 3\sqrt{x} < 180, x < 3600.$$

9. Suppose the demand function of a product is $p = \frac{1}{\sqrt{x}}$ and the supply function of this product is $p = 9\sqrt{x}$. Then the consumer's surplus of this product is

- (A) $\frac{1}{2}$; (B) $\frac{1}{3}$; (C) $\frac{1}{4}$; (D) 1; (E) $\frac{3}{2}$; (F) $\frac{2}{3}$.

Solution. (B) The equilibrium point is obtained by $\frac{1}{\sqrt{x}} = 9\sqrt{x}$, $x = \frac{1}{9}$, $p = 3$. The consumers surplus is $\int_0^{1/9} \left(\frac{1}{\sqrt{x}} - 3 \right) dx = \left[2\sqrt{x} - 3x \right]_{x=0}^{1/9} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$.

10. Suppose the demand function of a product is $p = \frac{x}{2x+3}$. If the product level is increasing at a rate 4.5 units per month, what is the rate of change of the price (in dollars per month) when the price is $p = \$0.4$ per unit?

- (A) 0.05; (B) 0.025; (C) 0.02; (D) 0.01; (E) 0.015; (F) 0.03.

Solution. (C) The price p and the production level x are two functions of time t , which are related by the equation given in the problem. Take the derivative of this equation on both sides with respect to t . $p' = \frac{x'(2x+3) - x(2x')}{(2x+3)^2} = \frac{3x'}{(2x+3)^2}$. When $p = 0.4$, solving the equation

$0.4 = \frac{x}{2x+3}$ for x , we have $x = 6$. Now $x' = 4.5$, $x = 6$. Plug these values into the relation of the derivatives, we have $p' = 0.02$, i.e., the price is increasing at a rate \$0.02 per month.

Section B. Long Answer Questions (30 marks)

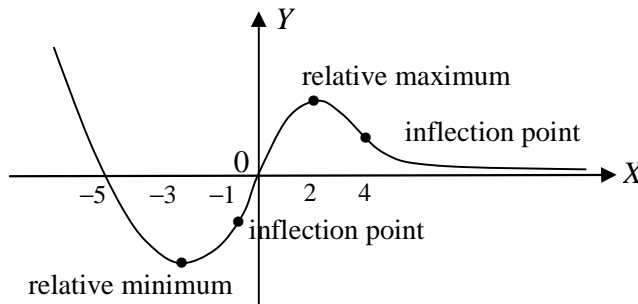
1. (6 marks) Sketch the graph of a continuous function $y = f(x)$, $-\infty < x < \infty$, that satisfies all of the following conditions:

- (a) $f(-5) = f(0) = 0$.
 (b) $f'(x) < 0$ when $x < -3$ or $x > 2$, and $f'(x) > 0$ when $-3 < x < 2$.
 (c) $f''(x) > 0$ when $x < -1$ or $x > 4$, and $f''(x) < 0$ when $-1 < x < 4$.

(d) $\lim_{x \rightarrow \infty} f(x) = 0$.

Mark relative maxima, relative minima, and inflection points if any.

Solution.



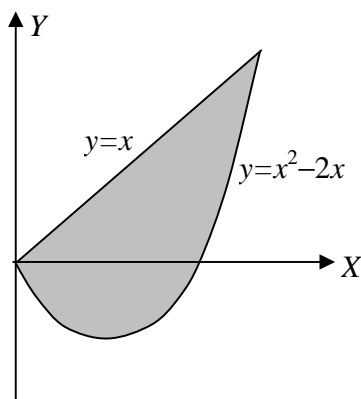
2. (6 marks) (a) (1 mark) Sketch the region under the line $y = x$ above the parabola $y = x^2 - 2x$.

(b) (1 mark) Find the intersection points of the line $y = x$ and parabola $y = x^2 - 2x$.

(c) (2 marks) Construct a definite integral that gives area enclosed by the graphs of functions $y = x^2 - 2x$ and $y = x$.

(d) (2 marks) Evaluate this definite integral.

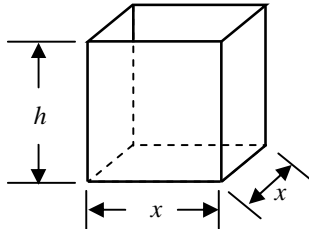
Solution. (a) The area is shown as the following



(b) Let $x^2 - 2x = x$. $x^2 - 3x = 0$. $x = 0, 3$.

(c), (d) The area is $\int_0^3 (x - x^2 + 2x) dx = \left[-\frac{x^3}{3} + \frac{3x^2}{2} \right]_{x=0}^3 = \frac{9}{2}$.

3. (6 marks) A company wants to make a rectangular open top box with a square base. The capacity of the box should be 6 liters (i.e., 6000 cm^3) as shown in the following figure:



The material used to make four vertical sides costs 0.2 cent per cm^2 , and the material to make the bottom costs 0.3 cents per cm^2 . Let the side length of the bottom be x , and let the height of the box be h , then the cost of the material to make this box is $C = 0.3x^2 + 0.2(4xh)$. Find the dimension of the box that minimizes the cost and the minimum cost.

Solution. Since the capacity of the box is $hx^2 = 6000$, $h = 6000/x^2$. The cost of the material that makes the can is $C = 0.3x^2 + 0.8xh = 0.3x^2 + \frac{4800}{x}$, $x > 0$. Let $C' = 0.6x - 4800/x^2 = 0$, $x^3 = 8000$, $x = 20$ cm. Then $h = 15$ cm. Since $C' < 0$ when $x < 20$, and $C' > 0$ when $x > 20$. When x approaches 0 or x approaches infinity, $C(x)$ approaches infinity. Therefore, function $C(x)$ attains an absolute minimum at $x = 20$ cm and $h = 15$ cm. The minimum cost is $0.3 \times 20^2 + 0.8 \times 20 \times 15 = \3.60 .

4. (6 marks) Find the following integrals:

(a) $\int_0^1 \frac{1}{1+\sqrt{x}} dx$.

(b) $\int x \ln x dx$.

Solution. (a) Let $u = 1 + \sqrt{x}$. $u' = \frac{1}{2\sqrt{x}}$. When $x = 0$, $u = 1$; when $x = 1$, $u = 2$.

$$\int_0^1 \frac{1}{1+\sqrt{x}} dx = \int_1^2 \frac{1}{1+\sqrt{x}} (2\sqrt{x}) du = 2 \int_1^2 \frac{u-1}{u} du = 2[u - \ln u]_{u=1}^2 = 2(1 - \ln 2).$$

(b) Use integration by parts:

$$\int x \ln x dx = \int \ln x d\left(\frac{x^2}{2}\right) = \frac{x^2 \ln x}{2} - \int \left(\frac{x^2}{2}\right) d \ln x = \frac{x^2 \ln x}{2} - \frac{1}{2} \int x dx = \frac{1}{2} x^2 \ln x - \frac{x^2}{4} + C.$$

5. (6 marks) Consider two variable function $z = 6xy + x^2 - y^3$.

- (a) (2 mark) Find the partial derivatives z_x and z_y .
- (b) (2 marks) Find all critical points of the function.
- (c) (2 marks) Determine whether each of the critical points is a relative maximum/minimum or a saddle point.

Solution. (a) $z_x = 6y + 2x$, $z_y = 6x - 3y^2$.

(b) Let $z_x = 0$, $z_y = 0$, we have $x = -3y$ and $2x = y^2$.

Hence, $-6y = y^2$. $y = 0, -6$. When $y = 0$, $x = 0$; when $y = -6$, $x = 18$.

Critical points are $(0, 0)$, $(18, -6)$.

(c) $z_{xx} = 2$, $z_{yy} = -6y$, $z_{xy} = 6$. $D = -12y - 36$.

At point $(0, 0)$, $D = -36 < 0$. $(0, 0)$ is a saddle point.

At point $(18, -6)$, $D = 36 > 0$, and $z_{xx} > 0$, z attains a relative minimum at this point.