Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et statistique

Faculty of Science
Mathematics and Statistics

MAT 1339 C
Calculus and Vectors
Practice Exam A
December 5th, 2012

Instructor: Thu Huong Nguyen

Duration: 3 hours

Name: SOLUTIONS

Student Number:________________________

Instructions:

- Print your name and student number on this page.
- Verify that your copy of the exam has all 12 pages.
- You must answer all questions. There are 10 Multiple Choice Questions and 7 Problems worth a total of 50 marks
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary.
- **No Notes or Books.**
- **Basic scientific calculators only – graphing and/or programmable calculators are NOT permitted.**

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Multiple Choice Questions (1 mark each)

Question 1 The limit $\lim_{x\to -1} \sqrt[3]{x^2+1}$ is?
(a) does not exist
(b) 0
(c) 1
(d) $\sqrt[3]{2}$

$\lim_{x\to -1} \sqrt[3]{x^2+1} = \sqrt{(-1)^2+1} = \sqrt[3]{2}$

Question 2 If the cost, in dollars, of producing x DVD is

$$C(x) = -0.004x^2 + 9.2x + 5000,$$

what is the true cost of producing the 1001st DVD (correct to the nearest number)?
(a) \$1
(b) \$2
(c) \$3
(d) \$4

$\Delta C = C(1001) - C(1000)$
$= -0.004(1001)^2 + 9.2(1001) + 5000 - (-0.004(1000)^2 + 9.2(1000) + 5000)$
≈ 1.196

Question 3 A radioactive isotope has a half-life of 3 years. How much of 10 grams of this substance is left after t years ?
(a) $m(t) = 10e^{\frac{t(\ln 2)}{3}}$
(b) $m(t) = 10e^{\frac{-t(\ln 2)}{3}}$
(c) $m(t) = 3e^{\frac{t(\ln 2)}{10}}$
(d) $m(t) = 3e^{\frac{-t(\ln 2)}{10}}$

Question 4 Determine where the graph of $f(x) = e^{-x^2}$ is concave up
(a) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
(b) $(-8, 0)$
(c) $\left(-\infty, -\frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, \infty\right)$
(d) $(0, \infty)$

$f'(x) = -2xe^{-x^2}$
$f''(x) = (-2+4x^2)e^{-x^2}$
$-2+4x^2 > 0$
$x < -\frac{\sqrt{2}}{2}$ or $x > \frac{\sqrt{2}}{2}$

Question 5 The critical points of the function $f(x) = x - \ln x^2$ are
(a) $x = 0$
(b) $x = 1$
(c) $x = 2$
(d) $x = 3$

$f' = 1 - \frac{2x}{x^2} = 1 - \frac{2}{x} = \frac{x-2}{x}$
$f'(x) = 0$ if $x - 2 = 0 \Rightarrow x = 2$

Question 6 Which of the following is FALSE?
(a) $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$
(b) The projection of $\vec{v}$ on $\vec{u}$ is the vector given by

$$\text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{v} \cdot \vec{v}}\right)\vec{v}.$$

(c) The volume of parallelepiped formed by $\vec{u}$, $\vec{v}$ and $\vec{w}$ is

$$V = |\vec{w} \cdot (\vec{u} \times \vec{w})|$$

(d) $\cancel{\vec{u} \times \vec{v} = \vec{v} \times \vec{u}}$ $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

Question 7 What is the magnitude of the torque of a force of 60 N applied at angle of 80° to a lever at a point that is 20 cm from the pivot?
(a) 11 Nm
(b) 11.8 Nm
(c) 12.5 Nm
(d) 12.8 Nm

$|\vec{\tau}| = |\vec{F}||\vec{r}|\sin(80^\circ)$
$= (60)(0.2)\sin(80) \approx 11.8$ Nm

Question 8 The vector equation of the line perpendicular to the plane

$$x + y - 10z + 2 = 0$$

and including the point $M(-3, 0, 1)$ is
(a) $[x, y, z] = [-3, 0, 1] + t[1, 1, 10]$
(b) $[x, y, z] = [-3, 0, 1] + t[1, 1, -10]$
(c) $[x, y, z] = [-3, 0, 1] - t[1, 1, 10]$
(d) $[x, y, z] = [-3, 0, 1] - t[-1, 1, 10]$

Question 9 Which of the following is FASLE ?
(a) There exist two lines which have exactly one point of intersection.
(b) There exist two lines which have exactly two points of intersection.
(c) There exist two lines which intersect at infinitely many points.
(d) There exist two lines with no intersection.

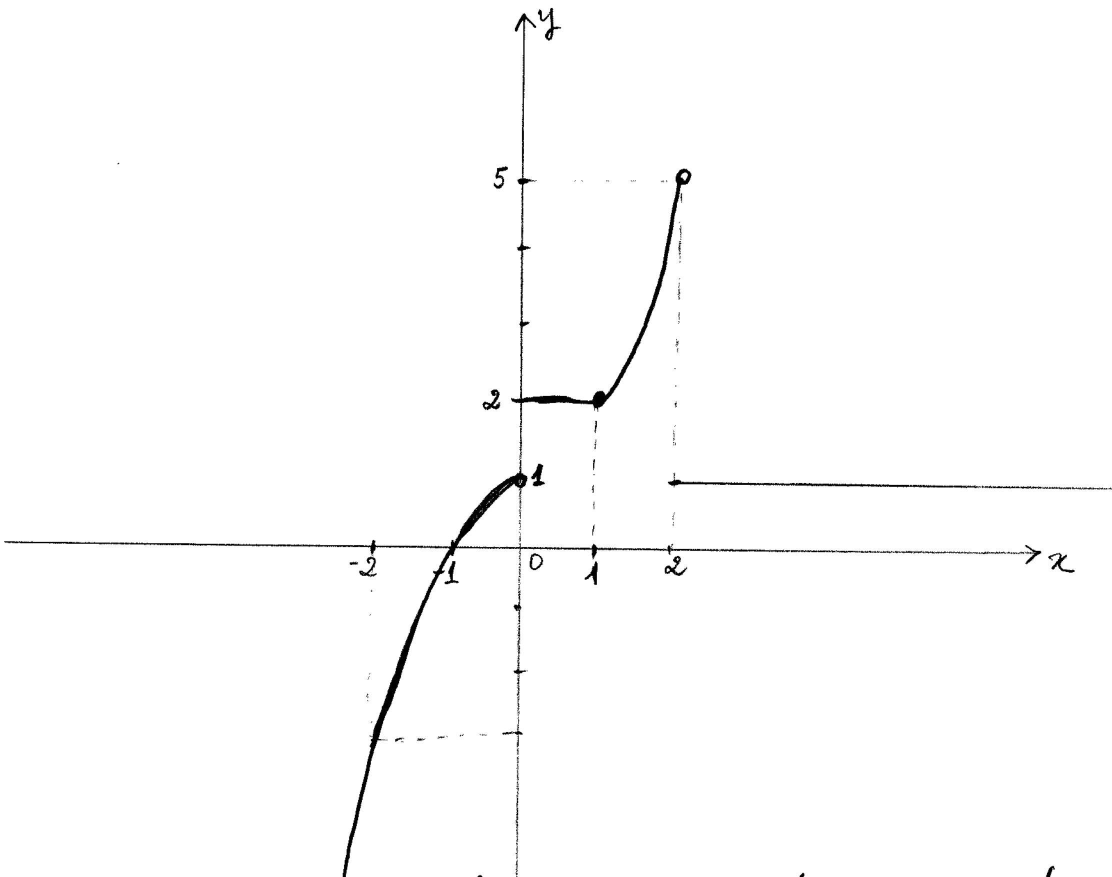
Question 10 Find m such that a plane $(P) : 3mx - 2y + z + 3 = 0$ and $(\ell) : [x, y, z] = [1, 2, 3] + t[1, -m, 5]$ are parallel.
(a) $m = -1$
(b) $m = 0$
(c) $m = 1$
(d) $m = 2$

$3m(1) + (-2)(-m) + (1)(5) = 0$
$3m + 2m + 5 = 0$
$5m + 5 = 0$
$m = -1$

Problems: (40 marks) Write complete solutions in the spaces provided.

Question 1 (? marks) Consider the function $f(x) = \begin{cases} -x^2+1 & x<0 \\ 2 & 0 \le x < 1 \\ x^2+1 & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$.

Determine if the function has any discontinuities. If so, what type(s) ?



There are two discontinuities at $x=0$ and $x=2$

They are jump discontinuities.

Question 2 (? marks) Let $y = f(x) = \dfrac{3x}{x^2+5}$.

(a) Determine the domain of $f(x)$ and the vertical asymptote(s) (if any) of $f(x)$.

$D = \mathbb{R}$, no vertical assymptote

(b) Determine the horizontal asymptote(s) of $f(x)$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x}{x^2+5} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(\frac{3}{x}\right)}{\cancel{x^2}\left(1+\frac{5}{x^2}\right)} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{3x}{x^2+5} = \lim_{x\to-\infty} \frac{\cancel{x^2}\left(\frac{3}{x}\right)}{\cancel{x^2}\left(1+\frac{5}{x^2}\right)} = 0$$

Answer: $y = 0$

(c) Find the intercepts of $f(x)$.

x-intercept: $y = 0$ if $\dfrac{3x}{x^2+5} = 0 \Rightarrow 3x = 0 \Rightarrow x = 0 \Rightarrow (0,0)$

y-intercept: $x = 0$ then $\dfrac{3(0)}{0^2+5} = 0 \Rightarrow (0,0)$

(d) Find the critical points (if any) and the intervals where $f(x)$ is increasing, decreasing.

$$y' = \frac{3(x^2+5) - 3x(2x)}{(x^2+5)^2} = \frac{-3x^2+15}{(x^2+5)^2}$$

$y' = 0$ if $-3x^2+15 = 0 \rightarrow x^2 = 5 \Rightarrow x = \pm\sqrt{5}$

CP : at $x = \sqrt{5}$ and $x = -\sqrt{5}$

x	$-\infty$		$-\sqrt{5}$		$\sqrt{5}$		∞
y'		$-$	0	$+$	0	$-$	
y	0	$\searrow$	L.Min	$\nearrow$	L.Max	$\searrow$	0

Increasing $(-\sqrt{5}, \sqrt{5})$

Decreasing : $(-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$

L. Max : $\left(\sqrt{5}, \frac{3\sqrt{5}}{10}\right)$; L Min $\left(-\sqrt{5}, \frac{-3\sqrt{5}}{10}\right)$

(e) Find the inflection points (if any) and the intervals where $f(x)$ is concave up, concave down.

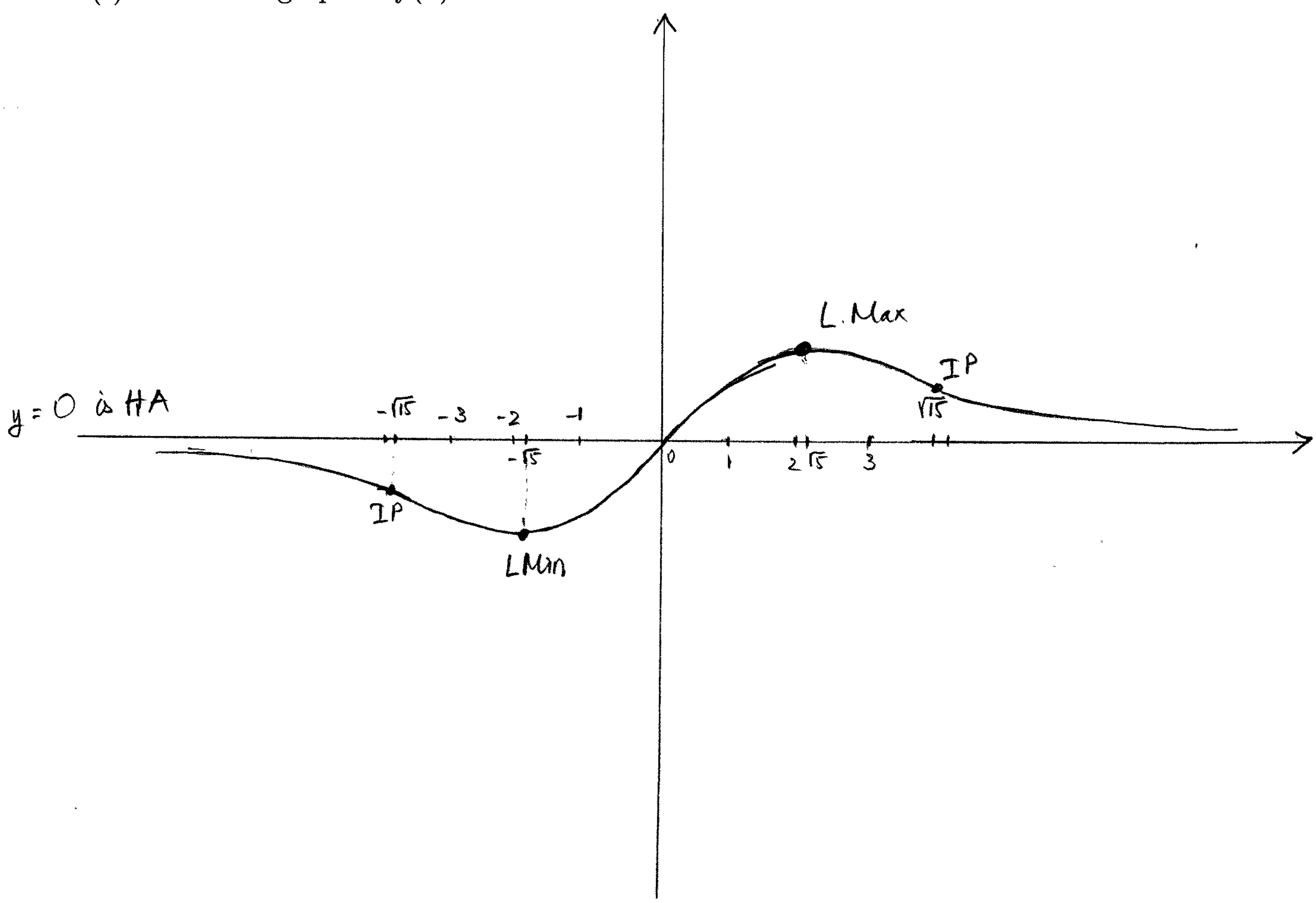
$$y'' = \frac{(-3x^2+15)'(x^2+5)^2 - (-3x^2+15)[(x^2+5)^2]'}{(x^2+5)^4}$$

$$y'' = \frac{6x^3-90x}{(x^2+5)^3} = \frac{6x(x^2-15)}{(x^2+5)^3}$$

$y'' = 0$ if $6x(x^2-15)=0$, then $\left[\begin{array}{l} x=0 \\ x=\pm\sqrt{15} \end{array}\right. \Leftarrow$ IP

x	$-\infty$		$-\sqrt{15}$		0		$\sqrt{15}$		∞
y''		$-$	0	$+$	0	$-$	0	$+$	
y		down $\cap$		up $\cup$		down $\cap$		up $\cup$	

(f) Sketch the graph of $f(x)$.



Question 3 (? marks) Find the derivatives of the following functions. Do not simplify:

(a) $f(x) = 5x^3 - e^{3x^2-5x} - 7\sqrt{x^2+3} + 2\sin(x^2+x-1) - 3\ln(4x^2 - \cos x)$

$$f'(x) = 15x^2 - e^{3x^2-5x}(3x^2-5x)' - \frac{7}{2\sqrt{x^2+3}}(x^2+3)'$$

$$+ 2\cos(x^2+x-1)(x^2+x-1)' - 3\,\frac{1}{4x^2-\cos x}(4x^2-\cos x)'$$

$$= 15x^2 - (6x-5)e^{3x^2-5x} - \frac{7(2x)}{2\sqrt{x^2+3}} + 2(2x+1)\cos(x^2+x-1)$$

$$- \frac{3(8x+\sin x)}{4x^2-\cos x}$$

(b) $f(x) = (x^4 - \frac{1}{x} + 2x^2 - \cos x)e^{\ln x}$ $\quad (uv)' = u'v + uv'$

$$f'(x) = \left(x^4 - \frac{1}{x} + 2x^2 - \cos x\right)' e^{\ln x} + \left(x^4 - \frac{1}{x} + 2x^2 - \cos x\right)\left(e^{\ln x}\right)'$$

$$= \left(4x^3 + \frac{1}{x^2} + 2 + \sin x\right)e^{\ln x} + \left(x^4 - \frac{1}{x} + 2x^2 - \cos x\right)e^{\ln x}(\ln x)'$$

$$= \left(4x^3 + \frac{1}{x^2} + 2 + \sin x\right)e^{\ln x} + \left(x^4 - \frac{1}{x} + 2x^2 - \cos x\right)e^{\ln x}\left(\frac{1}{x}\right)$$

(c) $f(x) = \dfrac{\cos(x^3) + \ln(x^2-1)}{e^{2x}+1}$ $\quad \left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

$$f'(x) = \frac{[\cos(x^3) + \ln(x^2-1)]'(e^{2x}+1) - [\cos(x^3) + \ln(x^2-1)](e^{2x}+1)'}{(e^{2x}+1)^2}$$

$$= \frac{\left[-\sin(x^3)(x^3)' + \frac{1}{x^2-1}(x^2-1)'\right](e^{2x}+1) - [\cos(x^3) + \ln(x^2-1)](2e^{2x})}{(e^{2x}+1)^2}$$

$$= \frac{\left[-3x^2\sin(x^3) + \frac{2x}{x^2-1}\right](e^{2x}+1) - 2e^{2x}\left(\cos(x^3) + \ln(x^2-1)\right)}{(e^{2x}+1)^2}$$

Question 4 (? marks) Let $\vec{u} = [1, 1, 9]$, $\vec{v} = [0, 0, 4]$ and $\vec{w} = [-2, 0, 5]$. Let θ be the angle between $\vec{u}$ and $\vec{v}$.

(a) Find the dot product $\vec{u} \cdot \vec{v}$.

$$\vec{u}\cdot\vec{v} = [1,1,9]\cdot[0,0,4] = (1)(0)+(1)(0)+(9)(4)$$
$$= \boxed{36}$$

(b) Find the cross product $\vec{u} \times \vec{v}$.

$$\vec{u}\times\vec{v} = [1,1,9]\times[0,0,4]$$
$$= [4, -4, 0]$$

$$\begin{array}{ccccc} 1 & 1 & 9 & 1 & 1 \\ 0 & 0 & 4 & 0 & 0 \end{array}$$

$(1)(4) - (9)(0) = 4$

$(9)(0) - (1)(4) = -4$

$(1)(0) - (1)(0) = 0$

(c) Find $\cos\theta$ and $\sin\theta$.

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta \Rightarrow \cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

$$|\vec{u}| = \sqrt{1^2+1^2+9^2} = \sqrt{83}$$
$$|\vec{v}| = \sqrt{0^2+0^2+4^2} = 4$$
$$\Rightarrow \cos\theta = \frac{(36)}{\sqrt{83}\,(4)} = \boxed{\frac{9}{\sqrt{83}}}$$

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta \Rightarrow \sin\theta = \frac{|\vec{u}\times\vec{v}|}{|\vec{u}||\vec{v}|}$$

$$|\vec{u}\times\vec{v}| = \sqrt{4^2+(-4)^2+0^2} = 4\sqrt{2}$$

$$\Rightarrow \sin\theta = \frac{4\sqrt{2}}{\sqrt{83}\,(4)} = \boxed{\frac{\sqrt{2}}{\sqrt{83}}} \text{ or } \boxed{\sqrt{\frac{2}{83}}}$$

(d) Find the projection $\text{proj}_{\vec{u}}\vec{v}$ of the vector $\vec{v}$ on $\vec{u}$.

$$\text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$

$\vec{v}\cdot\vec{u} = \vec{u}\cdot\vec{v} = 36$ (as in (a))

$\vec{u}\cdot\vec{u} = |\vec{u}|^2 = (\sqrt{83})^2 = 83$ (as in (c))

$$\text{proj}_{\vec{u}}\vec{v} = \left(\frac{36}{83}\right)[1, 1, 9]$$

(e) Find the volume of the parallelepiped formed by $\vec{u}$, $\vec{v}$ and $\vec{w}$.

$V = |\vec{w}\cdot(\vec{u}\times\vec{v})|$ $\qquad \vec{u}\times\vec{v} = [4, -4, 0]$ in (b)

$$\vec{w}\cdot(\vec{u}\times\vec{v}) = [-2, 0, 5]\cdot[4, -4, 0]$$
$$= (-2)(4) + (0)(-4) + (5)(0)$$
$$= -8$$

$\Rightarrow V = |-8| = 8$

Question 5 (? marks) Consider the plane (P) with scalar equation

$$3x + y - 2z + 15 = 0$$

(a) Determine if the point $Q(1, -1, 2)$ is on the plane (P).

Substitute the point Q into plane (P)

$$3(1) + (-1) - 2(2) + 15$$
$$3 - 1 - 4 + 15 = 13 \neq 0$$

Therefore, Q is not on the plane (P)

(b) Write the parametric equation of the line (ℓ), which is perpendicular to the plane (P) and passes through the point $M(-1, -2, -2)$.

$(\ell) \perp (P)$ so $\vec{d_\ell} = \vec{n_P} = [3, 1, -2]$

Parametric Eq. :

$$x = -1 + 3t$$
$$y = -2 + t$$
$$z = -2 - 2t$$

(c) Find the intersection of the line (ℓ) and the plane (P).

Substitute x, y, z of (ℓ) into (P)

$$3(-1+3t) + (-2+t) - 2(-2-2t) + 15 = 0$$

$$\Leftrightarrow \quad -3 + 9t - 2 + t + 4 + 4t + 15 = 0$$

$$\Leftrightarrow \quad 14t + 14 = 0 \Rightarrow t = -1$$

Then
$$x = -1 + 3(-1) = -4$$
$$y = -2 + (-1) = -3$$
$$z = -2 - 2(-1) = 0$$

$\Rightarrow$ Intersection $N(-4, -3, 0)$

(d) Determine the distance between the point $M(-1, -2, -2)$ and the plane (P).

Distance between the point $M(-1, -2, -2)$ and the plane (P) is MN

$$d = |\overrightarrow{MN}| = \sqrt{(-3)^2 + (-1)^2 + (2)^2} = \sqrt{14}$$

where $\overrightarrow{MN} = [-3, -1, 2]$

Question 6 (? marks) Consider the plane with direction vectors $\vec{a} = [2,3,4]$ and $\vec{b} = [1,0,3]$ throught the point $Q(0,1,-3)$.

(a) Write the vector and parametric equations of the plane.

Vector Eq.:

$$[x, y, z] = [0, 1, -3] + t[2, 3, 4] + s[1, 0, 3]$$

Parametric Eq:

$$x = 2t + s$$
$$y = 1 + 3t$$
$$z = -3 + 4t + 3s$$

(b) Write a scalar equation of the plane.

Normal vector of the plane (P)

$$\vec{n} = \vec{a} \times \vec{b} = [9, -2, -3]$$

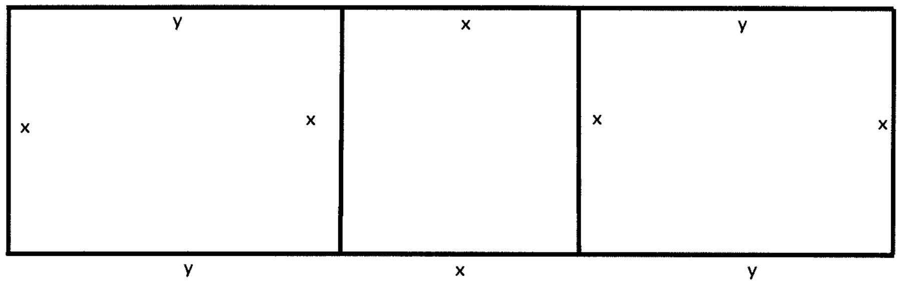
$$\begin{array}{ccccc} 2 & 3 & 4 & 2 & 3 \\ 1 & 0 & 3 & 1 & 0 \end{array}$$

$$(3)(3) - (0)(4) = 9$$
$$(4)(1) - (2)(3) = 4 - 6 = -2$$
$$(2)(0) - (3)(1) = -3$$

So: Scalar Eq:

$$9(x-0) - 2(y-1) - 3(z-(-3)) = 0$$
$$9x - 2y - 3z + 2 - 9 = 0$$
$$\boxed{9x - 2y - 3z - 7 = 0}$$

Question 7 (? marks) A farmer has 1200 meters of fencing to make a rectangular pen. The pen is to be divided into 3 sections by two fences running parallel to one of the sides. The middle section is square, and the others are equal rectangular, as shown. Find the dimensions that produce the pen of maximum area.



- Area of the pen

$$A = (\text{length}) \times (\text{width})$$
$$= (x+2y)(x) = x^2 + 2xy$$

- Condition :

Total length of fencing $= 1200 = 2(x+2y) + 4x$
$$\Leftrightarrow \quad 1200 = 6x + 4y$$
$$\Leftrightarrow \quad 3x + 2y = 600$$
$$\Rightarrow \quad y = \frac{600-3x}{2}$$

- Bounds of x : $0 < x < 200$

- $A = x^2 + 2xy = x^2 + 2x\left(\frac{600-3x}{2}\right)$
$$= -2x^2 + 600$$
$A' = -4x + 600$, $A' = 0$ if $x = 150$

$A'' = -4 < 0$, so A has local maximum at $x = 150$ and it is also the maximum we need to find.

$x = 150 \Rightarrow y = \frac{600 - 3(150)}{2} = 75$

Ans : $\boxed{x = 150 \ , \ y = 75}$