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Question 1. (4 points) A moving object has position vector

$$\vec{r}(t) = (9t^2 - 18t + 5)\,\hat{\imath} + (2t^3 - 6t)\,\hat{\jmath} + (3t^2 - 6t - 2)\,\hat{k}\,.$$

(a) Find the velocity and acceleration vectors.

(b) Are there any times at which the object is at rest? If yes, what is the acceleration at those times?

a) $\boxed{\vec{v}(t) = \vec{r}\,'(t) = (18t - 18)\hat{\imath} + (6t^2 - 6)\hat{\jmath} + (6t - 6)\hat{k}}$

$\boxed{\vec{a}(t) = \vec{v}\,'(t) = 18\hat{\imath} + 12t\hat{\jmath} + 6\hat{k}}$

b) $\vec{v}(t) = \vec{0}$ if $\boxed{t = 1}$

$\boxed{\vec{a}(1) = 18\hat{\imath} + 12\hat{\jmath} + 6\hat{k}}$

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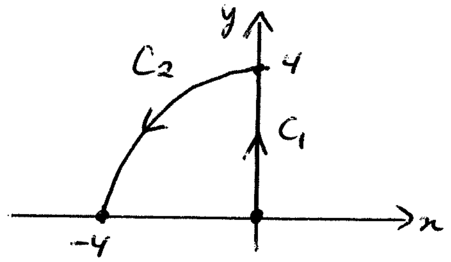
Question 2. (5 points) Calculate the line integral of $\vec{F}(x,y) = 2x\,\hat{\imath} + (1-2y)\,\hat{\jmath}$ along the path C, where C consists of the straight line from $(0,0)$ to $(0,4)$ and then a circular arc (part of the circle $x^2+y^2=16$) directly from $(0,4)$ to $(-4,0)$.



C_1 is $x=0,\ y=t \quad 0 \le t \le 4$

$\vec{r}(t) = t\hat{\jmath},\quad \vec{r}\,'(t) = \hat{\jmath}$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^4 (1-2t)\hat{\jmath}\cdot\hat{\jmath}\,dt = \int_0^4 (1-2t)\,dt = t - t^2\Big|_0^4 = \boxed{-12}$$

C_2 is $x = -4\sin t,\ y = 4\cos t \quad 0 \le t \le \pi/2$

$\vec{r}(t) = -4\sin t\,\hat{\imath} + 4\cos t\,\hat{\jmath},\quad \vec{r}\,'(t) = -4\cos t\,\hat{\imath} - 4\sin t\,\hat{\jmath}$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} (-8\sin t\,\hat{\imath} + (1-8\cos t)\hat{\jmath})\cdot(-4\cos t\,\hat{\imath} - 4\sin t\,\hat{\jmath})\,dt$$

$$= \int_0^{\pi/2} (64\sin t\cos t - 4\sin t)\,dt$$

$$= 32\sin^2 t + 4\cos t\Big|_0^{\pi/2} = (32+0)-(0+4) = \boxed{28}$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} = 28-12 = \boxed{16}$$

OR easier way: since $\vec{F} = 2x\hat{\imath} + (1-2y)\hat{\jmath} = \nabla f$ where $f(x,y) = x^2+y-y^2$, $\vec{F}$ is path-independent and we can choose any path from $(0,0)$ to $(-4,0)$, like the straight line $\vec{r}(t) = -t\,\hat{\imath},\ 0\le t\le 4,\ \vec{r}\,'(t) = -\hat{\imath}$

so $$\int_C \vec{F}\cdot d\vec{r} = \int_0^4 (-2t\,\hat{\imath})\cdot(-\hat{\imath})\,dt = \int_0^4 2t\,dt = t^2\Big|_0^4 = \boxed{16}$$

OR easiest way: Since $\vec{F} = \nabla f$, where $f(x,y) = x^2+y-y^2$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(-4,0) - f(0,0) = (-4)^2 + 0 - 0 = \boxed{16}$$

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Question 3. (6 points) Exactly one of the following vector fields is a gradient field. Determine which one it is and explain why the others are not gradient vector fields. What is the potential function for the gradient vector field ?
$\vec{F}(x, y) = \sin(y)\,\hat{\imath} + \cos(y)\,\hat{\jmath}$
$\vec{G}(x, y) = xy^2\,\hat{\imath} - x^2\,\hat{\jmath}$
$\vec{H}(x, y) = (2x + \sin(y))\,\hat{\imath} + x\cos(y)\,\hat{\jmath}$

all of the vector fields are defined in all of $\mathbb{R}^2$, so we can check if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ or not

$\vec{F}(x,y) = \sin y\,\hat{\imath} + \cos y\,\hat{\jmath}$ $\quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - \cos y \neq 0$

not gradient field

$\vec{G} = xy^2\,\hat{\imath} - x^2\,\hat{\jmath}$ $\quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2xy \neq 0$

not gradient field

$\vec{H} = (2x + \sin y)\hat{\imath} + x\cos y\,\hat{\jmath}$ $\quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cos y - \cos y = 0$

so $\boxed{\vec{H} \text{ is gradient field}}$

$\vec{H} = \nabla h$ so $h(x,y) = \int (2x + \sin y)\,dx + g(y)$
$= x^2 + x\sin y + g(y)$

then $\frac{\partial h}{\partial y} = x\cos y + g'(y) = Q = x\cos y \Rightarrow g'(y) = 0$

so $g(y) = C$

$\therefore \boxed{h(x,y) = x^2 + x\sin y + C}$ is the potential function

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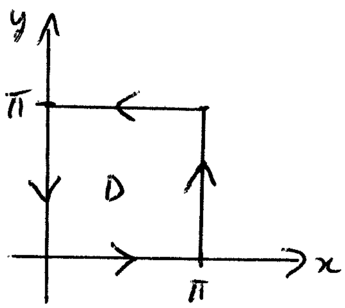
Question 4. (5 points) Let C be the square with corners at $(0,0)$, $(\pi,0)$, (π,π), and $(0,\pi)$, oriented counter-clockwise. If $\vec{F}(x,y) = x\cos(y)\,\hat{\imath} + y\sin(x)\,\hat{\jmath}$, calculate $\oint_C \vec{F}\cdot d\vec{r}$ using Green's Theorem.



Green's Theorem $\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$P(x,y) = x\cos y$

$Q(x,y) = y\sin x$

so $\oint_C \vec{F}\cdot d\vec{r} = \int_0^{\pi}\int_0^{\pi} (y\cos x + x\sin y)\,dx\,dy$

$= \int_0^{\pi} \left(y\sin x + \frac{1}{2}x^2\sin y \Big|_0^{\pi}\right) dy$

$= \int_0^{\pi} \frac{1}{2}\pi^2 \sin y\, dy$

$= \frac{1}{2}\pi^2 \left(-\cos y \Big|_0^{\pi}\right)$

$= \frac{1}{2}\pi^2 (2)$

$= \boxed{\pi^2}$