MATH2004 C — Test 1: 11:35 am - 12:25 pm, Sept. 28

Name and Student Number:

Total points: 20. No partial marks for Questions 1-4.

Closed book! Non-programmer calculators are allowed!

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1. (1.5 marks) Let $f(x) = \begin{cases} 1-x, & 0 \le x < 1; \\ -2-x, & 1 \le x < 2. \end{cases}$ Let $f_{odd}(x)$ be the 4-periodic **odd** extension of $f(x)$. Which of the following is the expression of $f_{odd}(x)$ when $-1 < x < 0$?

   (a) $-2-x$ (b) $1+x$ (c) $-1+x$ (d) $1-x$ (e) $-1-x$

   Solution: (e)
   $f_{odd}(x) = -f(-x) = -(1-(-x)) = -(1+x) = -1-x.$

2. (1.5 marks) Let $f(x) = \begin{cases} 1-x, & 0 \le x < 1; \\ 0, & 1 \le x < 2. \end{cases}$ Determine the value to which the Fourier sine series of $f(x)$ converges at $x = 11.7$.

   (a) 0.3 (b) -10.7 (c) 11.7 (d) 0 (e) -0.7

   Solution: (e)
   $B = \frac{f_{odd}(11.7+)+f_{odd}(11.7-)}{2} = f_{odd}(11.7) = f_{odd}(11.7-12) = f_{odd}(-0.3) = -1+0.3 = -0.7.$

3. (1.5 marks) Let $f(x) = \begin{cases} 1-x, & 0 \le x < 1; \\ 0, & 1 \le x < 2. \end{cases}$ Let a_n $(n = 0, 1, 2, ...)$ be the coefficients of the Fourier cosine series. Find a_2.

   (a) $\frac{2}{\pi^2}$ (b) $\frac{1}{\pi^2}$ (c) $\frac{1}{\pi}$ (d) $-\frac{1}{\pi}$ (e) 0

   Solution: (a).

$$
\begin{aligned}
a_2 &= \frac{2}{L}\int_0^L f(x)\cos(\frac{2\pi x}{L})\,dx = \frac{2}{2}\int_0^2 f(x)\cos(\frac{2\pi x}{2})\,dx \\
&= \int_0^1 (1-x)\cos(\pi x)\,dx \\
&= \left[\frac{1}{\pi}(1-x)\sin(\pi x) - \frac{1}{\pi^2}\cos(\pi x)\right]_0^1 \\
&= -\frac{1}{\pi^2}\cos(\pi) + \frac{1}{\pi^2} = \frac{2}{\pi^2}.
\end{aligned}
$$

4. (1.5 marks) Consider the parametric curve defined by $x = 1+3t^2+2t^3$, $y = 3t^2+2t^3-12t$. Find the slope of the tangent line at $t = 2$.

   (a) 1 (b) $\frac{1}{3}$ (c) $\frac{3}{2}$ (d) 0 (e) $\frac{2}{3}$

   Solution: (e)

$$
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t+6t^2-12}{6t+6t^2} = \frac{t+t^2-2}{t+t^2}.
$$

   At $t = 2$, $\frac{dy}{dx} = \frac{4}{6} = \frac{2}{3}$.

5. (5 points) The parametric curve C is given by $x = t - \frac{1}{2}t^2$, $y = \frac{4}{3}t^{3/2}$, where $0 \le t \le 1$.

Find the length L of the curve C.

Solution: We have $x'(t) = 1 - t$, $\quad y'(t) = (4/3)(3/2)t^{1/2} = 2\sqrt{t}$. The length is

$$\begin{aligned} L &= \int_0^1 \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt \qquad \text{[2 marks]} \\ &= \int_0^1 \sqrt{1 - 2t + t^2 + 4t}\,dt = \int_0^1 (1+t)\,dt = 3/2. \end{aligned}$$

6. (9 points) Let $f(x) = \begin{cases} 0, & \text{for } x \in [-\pi, 0); \\ 1, & \text{for } x \in [0, \pi). \end{cases}$ and let $f(x)$ be 2π-periodic. Find the Fourier series.

Solution:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0dx + \int_0^{\pi} 1dx\right) = 1, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cos(nx)dx + \int_0^{\pi} 1\cos(nx)dx\right) = \frac{1}{n\pi}\sin(nx)|_0^{\pi} = 0, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\sin(nx)dx + \int_0^{\pi} 1\sin(nx)dx\right) = -\frac{1}{n\pi}\cos(nx)|_0^{\pi} \\ &= \frac{1-(-1)^n}{n\pi} \quad \textit{or} \quad \frac{1+(-1)^{n+1}}{n\pi} \end{aligned}$$

Thus, the Fourier series is:

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin(nx) \quad \textit{or} \quad \frac{1}{2} + \sum_{\textit{odd } n} \frac{2}{n\pi}\sin(nx)$$

Marking: 2+3+3+1