# Math 3012 Assignment 1

## Solutions

#1) We hone $\vec{u} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{v} = \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$.

Then $\vec{u} + \vec{v} = \langle 2, -1, 1\rangle + \langle 1, -2, 2\rangle = \langle 3, -3, 3\rangle$,

$\vec{u} \cdot \vec{v} = \langle 2, -1, 1\rangle \cdot \langle 1, -2, 2\rangle = (2)(1) + (-1)(-2) + (1)(2) = 6$,

$\|\vec{u}\| = \|\langle 2, -1, 1\rangle\| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$,

$\|\vec{v}\| = \|\langle 1, -2, 2\rangle\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{1+4+4} = 3$,

and $\vec{u} \times \vec{v} = \langle 2, -1, 1\rangle \times \langle 1, -2, 2\rangle =$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 1 & -2 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} \hat{k}$$

$$= 0\hat{\imath} - 3\hat{\jmath} - 3\hat{k} = \langle 0, -3, -3\rangle.$$

#2) To show that the triangle is a right triangle, we will show that two of the sides are perpendicular,

Solutions

#2 continued

The vectors for the sides are $\vec{PQ} = \langle 2, -9, -1 \rangle$, $\vec{RP} = \langle 2, 4, 4 \rangle$ and $\vec{RQ} = \langle 4, -5, 3 \rangle$. We have

$$\vec{RP} \cdot \vec{RQ} = \langle 2,4,4 \rangle \cdot \langle 4,-5,3 \rangle = 8 - 20 + 12 = 0,$$

so these two sides are perpendicular.

#3) To find the equation for a sphere, we need to find its centre and its radius. Its centre is the midpoint of the line segment. This is $\langle \frac{4+2}{2}, \frac{2+4}{2}, \frac{0+6}{2} \rangle = \langle 3, 3, 3 \rangle$. Its radius is $\frac{1}{2}$ the length of the line segment. The length of the line segment is $\|\langle -2, 2, 6 \rangle\| = \sqrt{4+4+36} = \sqrt{44} = 2\sqrt{11}$. Thus, the equation of the circle is

$$(x-3)^2 + (y-3)^2 + (z-3)^2 = 44.$$

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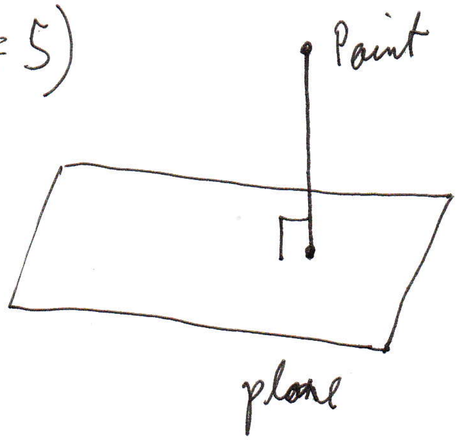
#4) To find an equation for a plane, we need a point on it and a normal vector. We are given that $(2,-1,3)$ lies on the plane. Since the given line is perpendicular to the plane, its direction vector, $\langle 3,-2,5\rangle$, is a normal vector. An equation for the plane is

$$3(x-2)-2(y+1)+5(x-3)=0.$$

#5)




The shortest distance from a point to a plane is along a line passing through the point and perpendicular to the plane.

A normal vector for the plane will give a direction vector for the line, so our line is $\vec{r}(t)=\langle 2,1,-5\rangle+t\langle 6,13,5\rangle$.

over

Solutions

#5 continued

Next, we need to find the point where this line meets the plane, i.e. we need to find the point that is on the line and on the plane. To do this, we plug $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ into the equation of the plane and solve for $t$:

$$6(2+6t) + 13(1+13t) + 5(-5+5t) = -4$$

$$12 + 36t + 13 + 169t - 25 + 25t = -4$$

$$230t = -4$$

$$t = \frac{-2}{115}$$

The point on the plane which is closest is therefore $\vec{r}\left(\frac{-2}{115}\right)$.

The distance from our point, $\vec{r}(0)$ to the plane is therefore

$$\left\| \vec{r}(0) - \vec{r}\left(\tfrac{-2}{115}\right) \right\| = \left\| \left(\frac{-2}{115}\right) \langle 6, 13, 5 \rangle \right\| = \frac{2}{115} \left\| \langle 6, 13, 5 \rangle \right\|$$

$$= \frac{2}{115} \sqrt{36 + 169 + 25} = \frac{2\sqrt{230}}{115}.$$

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#6) We have:

$$\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) - (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{v} + 2\vec{u}\cdot\vec{v}$$

$$= 4(\vec{u}\cdot\vec{v})$$ as required.

Using this, we have

$\vec{u}\perp\vec{v}$ iff $\vec{u}\cdot\vec{v}=0$

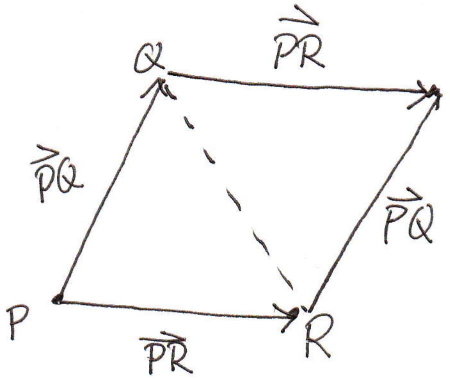
iff $\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 = 0$ (from above)

iff $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}-\vec{v}\|^2$

iff $\|\vec{u}+\vec{v}\| = \|\vec{u}-\vec{v}\|$,

as required.

#7)




The area of the triangle PQR is 1/2 the area of the parallelogram determined by $\vec{PQ}$ and $\vec{PR}$, and this is $\|\vec{PQ}\times\vec{PR}\|$.

over

Solutions

#7 continued

$$\text{Area} = \frac{1}{2}\|\vec{PQ}\times\vec{PR}\| = \frac{1}{2}\|\langle -2,-2,-3\rangle \times \langle 1,-3,-3\rangle\|.$$

$$\langle -2,-2,-3\rangle \times \langle 1,-3,-3\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & -2 & -3 \\ 1 & -3 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & -3 \\ -3 & -3 \end{vmatrix}\hat{\imath} - \begin{vmatrix} -2 & -3 \\ 1 & -3 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} -2 & -2 \\ 1 & -3 \end{vmatrix}\hat{k}$$

$= \langle -3, -9, 8\rangle$, so

$$\text{Area} = \frac{1}{2}\|\langle -3,-9,8\rangle\| = \frac{1}{2}\sqrt{9+81+64} = \frac{\sqrt{154}}{2}.$$

#8) To find the volume of a parallelpiped, we compute the scalar triple product of its edges, and take the absolute value.

ones

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#8 continued

$$\text{Volume} = \left| \begin{vmatrix} 3 & 0 & 1 \\ 3 & 2 & 5 \\ -1 & 0 & 3 \end{vmatrix} \right| = \left| (3)\begin{vmatrix} 2 & 5 \\ 0 & 3 \end{vmatrix} + 0 + (1)\begin{vmatrix} 3 & 2 \\ -1 & 0 \end{vmatrix} \right|$$

$$= \left| (3)(6) + (1)(2) \right| = |20| = 20.$$

#9) $\text{Work} = \vec{F} \cdot \vec{d} = \|\vec{F}\| \cdot \|\vec{d}\| \cos\theta =$

$(30 \text{ Newtons})(30 \text{ metres}) \cos 30^\circ = \dfrac{900\sqrt{3}}{2}$ Joules.

#10) By definition $\vec{\tau} = \vec{r} \times \vec{F} = \langle 1,1,1 \rangle \times \langle 2,2,1 \rangle$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} \hat{k}$$

$$= \langle -1, 1, 0 \rangle$$

and the magnitude of this is $\|\langle -1, 1, 0 \rangle\| = \sqrt{2}$.