CARLETON UNIVERSITY

FINAL/DEFERRED
EXAMINATION
December 2003

DURATION: 3 HOURS

Department Name & Course Number: Mathematics and Statistics MATH 1107EF
Course Instructors : Dr. A. Alaca, Dr. Z. Hu

AUTHORIZED MEMORANDA

NON PROGRAMMABLE CALCULATORS ARE ALLOWED

Students **MUST** count the number of pages in this examination question paper **before** beginning to write, and report any discrepancy immediately to a proctor. This question paper has **11** pages, including this top page.

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Please, check the appropriate box: FINAL : ☐ DEFERRED FINAL : ☐

ANSWER ALL QUESTIONS IN PART A and PART B (pp.2-10)

Good luck!

For departmental use only

PART A

Number of Correct Answers	Maximum Mark	Mark Obtained
	40	

PART B

Question	Maximum Mark	Mark Obtained
1	12	
2	12	
3	14	
4	14	
5	8	
TOTAL (PART A+PART B)	100	

TERM MARK : /40

EXAM MARK : /60

TOTAL MARK : /100

LETTER GRADE :

PART A: There are 16 multiple choice questions. Circle the correct answer. Each question is 2.5 marks. No partial marks. Total of 40 marks.

1. Let $A = \begin{bmatrix} 2 & -5 \\ -4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 5 \\ 4 & k \end{bmatrix}$.

For what value of k, $AB = BA$?

a) 5 **b)** 6 **c)** 7 **d)** 8

2. Let $A = \begin{bmatrix} k & 2 & 4 \\ 2 & 3 & 1 \\ 0 & k+2 & 0 \end{bmatrix}$. Find <u>all the values of k</u> such that $\det A = 0$.

a) 2 and 8 **b)** 2 and -8 **c)** -2 and 8 **d)** -2 and -8

3. Let $A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -6 & 2 & 3 \\ 4 & -1 & -2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$.

What is the second row of $(BA)^{-1}$?

a) $[1 \quad 6 \quad -1]$ **b)** $[-1 \quad -10 \quad 4]$ **c)** $[1 \quad 7 \quad -2]$ **d)** $[6 \quad -10 \quad 7]$

4. Let $T : R^4 \longrightarrow R^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 + x_2 - x_4 \\ x_2 - x_3 + 2x_4 \end{bmatrix}.$$

Which one of the following statements is TRUE?

a) T is one-to-one and onto.

b) T is one-to-one but not onto.

c) T is onto but not one-to-one.

d) T is neither one-to-one nor onto.

5. Let $T : R^2 \longrightarrow R^3$ be a linear transformation such that

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} -4 \\ 4 \end{bmatrix}\right)$=?

a) $\begin{bmatrix} -4 \\ -12 \\ 4 \end{bmatrix}$ **b)** $\begin{bmatrix} 12 \\ 4 \\ -12 \end{bmatrix}$ **c)** $\begin{bmatrix} 4 \\ 12 \\ -4 \end{bmatrix}$ **d)** $\begin{bmatrix} -12 \\ 4 \\ -4 \end{bmatrix}$

6. Let $\lambda = 3$ be an eigenvalue of a 4×4 matrix A.
Exactly one of the following statement is FALSE. Which one?

a) $\det(A - 3I) = 0$

b) Eigenspace corresponding to eigenvalue 3 is $\mathrm{Nul}(A - 3I)$.

c) The rank of the matrix $A - 3I$ is 4.

d) $Ax = 3x$ for some non-zero vectors x in R^4.

7. Let A be a 3×3 matrix with the characteristic polynomial $p(\lambda) = (\lambda + 1)(\lambda - 3)(\lambda + 4)$.
Exactly one of the following statement is FALSE. Which one?

a) Each eigenspace of A is one-dimensional.

b) A is invertible.

c) $A + 4I$ is invertible.

d) A is diagonalizable.

8. Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 5 \end{bmatrix}$. What is the distance between the vectors u and v?

a) $\sqrt{11}$ **b)** 6 **c)** $\sqrt{32}$ **d)** $\sqrt{35}$

9. What is the standard form of the complex number $\dfrac{2+3i}{2-3i}$?

a) $\frac{5}{13} - \frac{12}{13}i$ **b)** $\frac{5}{13} + \frac{12}{13}i$ **c)** $\frac{-5}{13} - \frac{12}{13}i$ **d)** $\frac{-5}{13} + \frac{12}{13}i$

10. If $z = 1 + i\sqrt{3}$, what is the standard form of z^{27}?

a) 2^{27} **b)** $-i2^{27}$ **c)** -2^{27} **d)** $i2^{27}$

11. What are the eigenvalues of the matrix $\begin{bmatrix} 0 & 13 \\ -1 & 6 \end{bmatrix}$?

a) $\lambda_1 = 3i$, $\lambda_2 = -3i$ **b)** $\lambda_1 = 2 + 3i$, $\lambda_2 = 2 - 3i$

c) $\lambda_1 = -1$, $\lambda_2 = 13$ **d)** $\lambda_1 = 3 + 2i$, $\lambda_2 = 3 - 2i$

12. Let $u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $u_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ and $u_5 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and

$H = \mathrm{Span}\{u_1, u_2, u_3, u_4, u_5\}$. What is the dimension of the subspace H?

a) 2 **b)** 3 **c)** 4 **d)** 5

13. If $\det A = 5$, $\det B = 3$ and $\det(A^T B^2 C) = 90$, what is $\det C$?

a) 2 **b)** 3 **c)** 6 **d)** 15

14. If $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1}$, what is A^{2003}?

a) $\begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix}$ **b)** $\begin{bmatrix} -5 & -4 \\ 6 & 5 \end{bmatrix}$ **c)** $\begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$ **d)** $\begin{bmatrix} 5 & 4 \\ -6 & -5 \end{bmatrix}$

15. What is the angle between the vectors $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -2 \end{bmatrix}$?

a) $\pi/2$ **b)** $\pi/3$ **c)** $\pi/4$ **d)** $\pi/6$

16. Which one of the following set of vectors form an orthogonal basis for R^3?

a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$ **b)** $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$ **d)** $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} \right\}$

<u>Part B:</u> There are 5 long questions of total 60 marks. Show all your work.

[12] **1.** You are given that $A = \begin{bmatrix} 1 & -4 & 4 & 5 \\ -1 & 5 & -3 & -4 \\ -1 & 7 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 4 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

a) Find a basis for Col$A$.

b) Find a basis for Nul$A$.

c) Find the dimension of Col$A$ and the dimension of Nul A.

d) Determine if the vector $\begin{bmatrix} -6 \\ 8 \\ 10 \end{bmatrix}$ is in the column space of A.
Explain your reason clearly.

[12] **2.** Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$.
You are given that characteristic polynomial of the matrix A is $P(\lambda) = \lambda^3(\lambda - 5)$.

a) Find a basis for each eigenspace of A.
b) Find, if possible, an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

[14] **3.** Let $u = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$, $v = \begin{bmatrix} -2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$, $x = \begin{bmatrix} 7 \\ 4 \\ -9 \\ 8 \end{bmatrix}$ and $W = \text{Span}\{u,\ v\}$.

a) Find the orthogonal projection of x onto W.

b) What is the closest point to x in W?

c) Find the distance from x to W.

d) Find a basis for the orthogonal complement of W, i.e find a basis for $W^{\perp}$.

e) Find an orthonormal basis for W.

[14] **4.a)** Let $H = \left\{ \begin{bmatrix} a+b \\ a-c \\ b+c \end{bmatrix} \middle|\ a,\ b,\ c \in R \right\}$.

Show that H is a subspace of R^3 and find a basis for H.

b) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 1 & 2 & 2 \end{bmatrix}$. Find the inverse of the matrix A.

c) Let $A = \begin{bmatrix} 4 & 1 & 1 & 2 \\ 0 & 0 & 8 & 0 \\ 2 & 1 & 2 & 2 \\ 5 & 1 & 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 0 \\ 7 \\ 4 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

You are given that $\det A = 16$.
Use Cramer's Rule to find x_3 in the matrix equation $Ax = b$.

[8] **5.a)** Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\}$ be a basis for R^3 and let $x = \begin{bmatrix} 1 \\ -7 \\ -3 \end{bmatrix}$.

Find the coordinate vector of x relative to basis $\mathcal{B}$, i.e., $[x]_{\mathcal{B}}$.

b) Let $u_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -2 \end{bmatrix}$, and $u_3 = \begin{bmatrix} -6 \\ -2 \\ 5 \\ -5 \end{bmatrix}$.

You are given that $S = \{u_1,\ u_2,\ u_3\}$ is an orthogonal set and that the vector $x = \begin{bmatrix} 3 \\ -19 \\ 19 \\ 5 \end{bmatrix}$ is in $\text{Span}\{u_1,\ u_2,\ u_3\}$. Find the value of c_1 in the equation $x = c_1u_1 + c_2u_2 + c_3u_3$.

Answers to multiple choice questions:
1.(d), 2.(c), 3.(b), 4.(c), 5.(c), 6(c), 7(c), 8.(d), 9(d),
10.(c), 11.(d), 12.(b), 13.(a), 14.(c), 15.(b), 16.(d).

Answers to long answer questions:

1. **(a)** $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 7 \end{bmatrix} \right\}$, **(b)** $\left\{ \begin{bmatrix} -9 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   (c) dimCol$A = 2$, dimNul$A = 2$.

   (d) $\left[\begin{array}{rr|r} 1 & -4 & -6 \\ -1 & 5 & 8 \\ -1 & 7 & 10 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -4 & -6 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{array}\right]$.

   The system is inconsistent. Hence the vector is not in Col$A$.

2. **(a)** A basis for E_0 is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$,

   A basis for E_5 is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

   (a) $P = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$.

3. **(a)** $\text{Proj}_W x = \hat{x} = \begin{bmatrix} 8 \\ 1 \\ -7 \\ 9 \end{bmatrix}$, **(b)** $\hat{x} = \begin{bmatrix} 8 \\ 1 \\ -7 \\ 9 \end{bmatrix}$, **(c)** $\text{dist}(x, W) = \sqrt{15}$.

   (d) A basis for $W^{\perp}$ is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

   (e) $\left\{ \begin{bmatrix} 1/\sqrt{15} \\ 2/\sqrt{15} \\ 1/\sqrt{15} \\ 3/\sqrt{15} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{15} \\ 1/\sqrt{15} \\ 3/\sqrt{15} \\ -1/\sqrt{15} \end{bmatrix} \right\}$.

4. **(a)** $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$. A basis for H is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

   (b) $A^{-1} = \begin{bmatrix} -10 & 2 & 11 \\ 4 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$. **(c)** $x_3 = 0$.

5. **(a)** $[x]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, **(b)** $c_1 = 6$.

Please let me know if you notice any typo in the answers.