

[10pts] 1. True (T) or false (F)? Circle the correct answer next to the statement.

(a) If the characteristic polynomial of an  $n \times n$  matrix  $A$  has  $n$  distinct roots, then  $A$  is diagonalizable.  T  F

(b) If a row echelon form of an  $n \times n$  matrix  $A$  has  $r$  zero rows, then the general solution of the homogeneous system of linear equations with coefficient matrix  $A$  has  $r$  parameters.  T  F

(c) If the feasible region of a linear programming problem is unbounded, then the problem has no optimal solution.  T  F

(d) The matrix  $A$  given below is diagonalizable.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

T  F

(e) A Markov process with transition matrix  $P$  given below possesses a steady state vector.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

T  F

2. For each of the following two questions, write your final answer in the answer box. *Show your work to receive partial marks.*

[4pts]

- (a) A wolf family lives in one of the three dens, A, B, or C. Once a week, the female may move her pups from one den to another. The following is observed. If the family is staying in den A or B, then the week after they will stay in the same den with 60% probability, and move to den C with 30% probability. However, if they are staying in den C, then the week after they will stay in den B with 70% probability, and are equally likely to move to one of the other two dens. Give the probability transition matrix for the Markov process described above.

$$A = \begin{bmatrix} 0.6 & 0.1 & 0.15 \\ 0.1 & 0.6 & 0.7 \\ 0.3 & 0.3 & 0.15 \end{bmatrix}$$

- (b) We have a sequence of numbers  $(x_0, x_1, x_2, \dots)$  that satisfies the recurrence relation  $x_{k+1} = 2x_k - 3x_{k-1}$  for  $k \geq 1$ . The initial terms are  $x_0 = 1$  and  $x_1 = 3$ . Set up a dynamical system of the form  $V_{k+1} = AV_k$  that will allow you to solve this recurrence relation; here  $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$ . That is, give the matrix  $A$  and initial vector  $V_0$ . *Do not solve the recurrence relation.*

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix} \quad V_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 2x_{k+1} - 3x_k \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$$

3. A baker brings two kinds of breads to the Organic Farmers' Market: sesame-spelt and gluten-free. She knows that she will be able to sell at most 100 loaves of bread, and that she should bring at least four times as many sesame-spelt loaves as gluten-free loaves. In addition, she remembers that she has a pre-order of 12 gluten-free loaves. If sesame-spelt loaves sell for \$4 a piece, and gluten-free loaves sell for \$6 a piece, how many of each kind of loaves should the baker bring to the market to maximize her revenue?

(a) Write down a mathematical model for this problem in the form of a standard (canonical) linear program. *Begin by defining your variables. Clearly indicate the objective function, the type of optimum you are seeking, and all constraints.*

[2pts]

$x = \#$  sesame-spelt loaves

$y = \#$  gluten-free loaves

LP:  $\max P = 4x + 6y$

subject to:  $x + y \leq 100$

$-x + 4y \leq 0$

$-y \leq -12$

$x, y \geq 0$

[2pts]

(b) Set up the initial simplex tableau for this problem. *Do not solve the problem.*

$x$	$y$	$s_1$	$s_2$	$s_3$	$P$	
1	1	1	0	0	0	100
-1	4	0	1	0	0	0
0	-1	0	0	1	0	-12
-4	-6	0	0	0	1	0

4. Consider the following simplex tableau for a linear program in standard form:

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$M$	
1	0	1	0	0	0	20
0	1	0	-1	0	0	12
1	0	0	2	1	0	30
-3	0	0	2	0	1	40

↑

[2pts] (a) Find the basic feasible solution associated with this tableau.

$$x_1 = x_4 = 0, \quad x_2 = 12, \quad x_3 = 20, \quad x_5 = 30, \quad M = 40$$

[4pts] (b) Use the simplex method to find an optimal solution to the problem starting from the tableau above. Clearly indicate the optimal values of the original variables and the objective function.

Bring  $x_1$  into the solution:

$$\frac{b_1}{a_{11}} = 20, \quad \frac{b_3}{a_{31}} = 30 \quad - \text{pivot on } a_{11}$$

Next tableau:

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$M$	
1	0	1	0	0	0	20
0	1	0	-1	0	0	12
0	0	-1	2	1	0	10
0	0	3	2	0	1	100

Basic feasible solution:  $x_3 = x_4 = 0$

$$x_1 = 20, \quad x_2 = 12, \quad x_5 = 10, \quad M = 100$$

Optimal solution:

$$x_1 = 20$$

$$x_2 = 12$$

$$M = 100 \quad \text{optimal value of the obj. function}$$

(2.5)

(1.5)

-0.5 if  
not

5. Consider a population with two age groups, senior and young. The population dynamics are described by the following equations:

$$s_{k+1} = \frac{3}{10}s_k + \frac{1}{5}y_k$$

$$y_{k+1} = \frac{11}{10}y_k$$

where  $s_k$  and  $y_k$  denote the sizes of the senior and young population, respectively, after  $k$  years. Write your final answers in the box below, and use the space at the bottom and on the next page for your work.

[10pts]

- (a) Suppose the initial sizes of the senior group and young group are  $s_0 = 20$  and  $y_0 = 40$ , respectively. What are the sizes of the two age groups after 1 year?

ANS:

senior group: 14  
young group: 44

- (b) Give an exact formula for the size of the senior group after  $k$  years (as a function of  $k$ ).

ANS:

$$s_k = 10 \left(\frac{3}{10}\right)^k + 10 \left(\frac{11}{10}\right)^k$$

- (c) Give a formula that approximates the size of the senior group after  $k$  years for  $k$  very large.

ANS:

$$s_k \approx 10 \left(\frac{11}{10}\right)^k \quad \text{for } k \text{ large}$$

- (d) Explain what happens with the size of the senior group in the long run.

ANS:

It grows without bound.

SPACE FOR WORK:

$$(a) \quad s_1 = \frac{3}{10}s_0 + \frac{1}{5}y_0 = \frac{3}{10} \cdot 20 + \frac{1}{5} \cdot 40 = 6 + 8 = 14$$

$$y_1 = \frac{11}{10}y_0 = \frac{11}{10} \cdot 40 = 44$$

SPACE FOR WORK (Question 5)

$$(b) \quad v_k = \begin{bmatrix} s_k \\ y_k \end{bmatrix}, \quad A = \begin{bmatrix} \frac{3}{10} & \frac{1}{5} \\ 0 & \frac{11}{10} \end{bmatrix}, \quad v_{k+1} = Av_k, \quad v_0 = \begin{bmatrix} 20 \\ 40 \end{bmatrix}$$

Diagonalize A:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - \frac{3}{10} & -\frac{1}{5} \\ 0 & \lambda - \frac{11}{10} \end{bmatrix} = \left(\lambda - \frac{3}{10}\right)\left(\lambda - \frac{11}{10}\right)$$

$$\lambda_1 = \frac{3}{10}, \quad \left[ \begin{array}{cc|c} 0 & -\frac{1}{5} & 0 \\ 0 & -\frac{8}{10} & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = \frac{11}{10}, \quad \left[ \begin{array}{cc|c} \frac{8}{10} & -\frac{1}{5} & 0 \\ 0 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array} \right] \quad x_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}, \quad P^{-1} = \frac{1}{4} \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \frac{3}{10} & 0 \\ 0 & \frac{11}{10} \end{bmatrix}$$

$$\begin{aligned} v_k = A^k v_0 &= P D^k P^{-1} v_0 = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \left(\frac{3}{10}\right)^k & 0 \\ 0 & \left(\frac{11}{10}\right)^k \end{bmatrix} \frac{1}{4} \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 40 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{3}{10}\right)^k & \left(\frac{11}{10}\right)^k \\ 0 & 4\left(\frac{11}{10}\right)^k \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 10\left(\frac{3}{10}\right)^k + 10\left(\frac{11}{10}\right)^k \\ 4\left(\frac{11}{10}\right)^k \end{bmatrix} \end{aligned}$$