[10pts] 1. True (T) or false (F)? *Circle the correct answer next to the statement.*

(a) If a row echelon form of an $n \times n$ matrix A has r leading ones, then the general solution of the homogeneous system of linear equations with coefficient matrix A has r parameters. T F

(b) If the feasible region of a linear programming problem is unbounded, then the problem has no optimal solution.

T F

(c) If the characteristic polynomial of an $n \times n$ matrix A has n distinct roots, then A is diagonalizable. T F

(d) A Markov process with transition matrix P given below possesses a steady state vector.

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{bmatrix}$$

T F

(e) The matrix A given below is not diagonalizable.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

T F

2. For each of the following two questions, write your final answer in the answer box. *Show your work to receive partial marks.*

[4pts]

(a) A wolf family lives in one of the three dens, A, B, or C. Once a week, the female may move her pups from one den to another. The following is observed. If the family is staying in den B or C, then the week after they will stay in the same den with 50% probability, and move to den A with 40% probability. However, if they are staying in den A, then the week after they will stay in den C with 80% probability, and are equally likely to move to one of the other two dens. Give the probability transition matrix for the Markov process described above.

$$A = \begin{bmatrix} 0.1 & 0.4 & 0.4 \\ 0.1 & 0.5 & 0.1 \\ 0.8 & 0.1 & 0.5 \end{bmatrix}$$

(b) We have a sequence of numbers $(x_0, x_1, x_2, \ldots)$ that satisfies the recurrence relation $x_{k+1} = -4x_k + 5x_{k-1}$ for $k \geq 1$. The initial terms are $x_0 = 0$ ad $x_1 = 2$. Set up a dynamical system of the form $V_{k+1} = AV_k$ that will allow you to solve this recurrence relation; here $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$. That is, give the matrix A and initial vector V_0. *Do not solve the recurrence relation.*

$$A = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} \qquad V_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ -4x_{k+1} + 5x_k \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$$

3. A baker brings two kinds of breads to the Organic Farmers' Market: sesame-spelt and gluten-free. She knows that she will be able to sell at most 90 loaves of bread, and that she should bring at least three times as many sesame-spelt loaves as gluten-free loaves. In addition, she remembers that she has a pre-order of 10 gluten-free loaves. If sesame-spelt loaves sell for \$5 a piece, and gluten-free loaves sell for \$7 a piece, how many of each kind of loaves should the baker bring to the market to maximize her revenue?

[2pts] (a) Write down a mathematical model for this problem in the form of a standard (canonical) linear program. *Begin by defining your variables. Clearly indicate the objective function, the type of optimum you are seeking, and all constraints.*

x = # sesame-spelt loaves
y = # gluten-free loaves

LP: max $p = 5x + 7y$ 0.5

subject to: $x + y \le 90$ 0.5

$-x + 3y \le 0$ 0.5

$-y \le -10$ 0.5

$x, y \ge 0$

[2pts] (b) Set up the initial simplex tableau for this problem. *Do not solve the problem.*

x	y	s_1	s_2	s_3	p	
1	1	1	0	0	0	90
-1	3	0	1	0	0	0
0	-1	0	0	1	0	-10
-5	-7	0	0	0	1	0

0.5 ea row

4. Consider the following simplex tableau for a linear program in standard form:

x_1	x_2	x_3	x_4	x_5	M	
0	1	0	1	0	0	15
1	0	−1	0	0	0	10
0	1	2	0	1	0	40
0	−3	2	0	0	1	20

[2pts] (a) Find the basic feasible solution associated with this tableau.

$X_2 = X_3 = 0, \quad X_1 = 10, \quad X_4 = 15, \quad X_5 = 40, \quad M = 20$

[4pts] (b) Use the simplex method to find an optimal solution to the problem starting from the tableau above. *Clearly indicate the optimal values of the original variables and the objective function.*

Bring x_2 into the solution:

$\frac{b_1}{a_{12}} = 15, \quad \frac{b_3}{a_{32}} = 40 \quad$ – pivot on a_{12}

Next tableau:

X_1	X_2	X_3	X_4	X_5	M	
0	1	0	1	0	0	15
1	0	−1	0	0	0	10
0	0	2	−1	1	0	25
0	0	2	3	0	1	65

Basic feasible solution: $X_3 = X_4 = 0$

$X_1 = 10, \; X_2 = 15, \; X_5 = 25, \; M = 65$

Optimal solution:

$X_1 = 10$

$X_2 = 15$

$M = 65$ optimal value of the obj. function

5. Consider a population with two age groups, senior and young. The population dynamics are described by the following equations:

$$s_{k+1} = \frac{2}{5}s_k + \frac{1}{10}y_k$$
$$y_{k+1} = \frac{6}{5}y_k$$

where s_k and y_k denote the sizes of the senior and young population, respectively, after k years. *Write your final answers in the box below, and use the space at the bottom and on the next page for your work.*

[10pts]

(a) Suppose the initial sizes of the senior group and young group are $s_0 = 40$ and $y_0 = 80$, respectively. What are the sizes of the two age groups after 1 year?

1

ANS: senior group : 24
young group : 96

(b) Give an exact formula for the size of the senior group after k years (as a function of k).

7

ANS: $s_k = 30\left(\frac{2}{5}\right)^k + 10\left(\frac{6}{5}\right)^k$

(c) Give a formula that approximates the size of the senior group after k years for k very large.

1

ANS: $s_k \cong 10\left(\frac{6}{5}\right)^k$ for k large

(d) Explain what happens with the size of the senior group in the long run.

1

ANS: It grows without bound

SPACE FOR WORK:

(a) $s_1 = \frac{2}{5}s_0 + \frac{1}{10}y_0 = \frac{2}{5}\cdot 40 + \frac{1}{10}80 = 16+8 = 24$

$y_1 = \frac{6}{5}y_0 = \frac{6}{5}\cdot 80 = 96$

SPACE FOR WORK (Question 5)

(b) $V_k = \begin{bmatrix} s_k \\ y_k \end{bmatrix}$ $A = \begin{bmatrix} \frac{2}{5} & \frac{1}{10} \\ 0 & \frac{6}{5} \end{bmatrix}$ $V_{k+1} = AV_k$, $V_0 = \begin{bmatrix} 40 \\ 80 \end{bmatrix}$

Diagonalize A:

$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - \frac{2}{5} & -\frac{1}{10} \\ 0 & \lambda - \frac{6}{5} \end{bmatrix} = \left(\lambda - \frac{2}{5}\right)\left(\lambda - \frac{6}{5}\right)$

$\lambda_1 = \frac{2}{5}$ $\left[\begin{array}{cc|c} 0 & -\frac{1}{10} & 0 \\ 0 & -\frac{4}{5} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = \frac{6}{5}$ $\left[\begin{array}{cc|c} \frac{4}{5} & -\frac{1}{10} & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{1}{8} & 0 \\ 0 & 0 & 0 \end{array}\right]$ $X_2 = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ 0 & 8 \end{bmatrix}$, $P^{-1} = \frac{1}{8}\begin{bmatrix} 8 & -1 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} \frac{2}{5} & 0 \\ 0 & \frac{6}{5} \end{bmatrix}$

$V_k = A^k V_0 = PD^kP^{-1}V_0 = \begin{bmatrix} 1 & 1 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} \left(\frac{2}{5}\right)^k & 0 \\ 0 & \left(\frac{6}{5}\right)^k \end{bmatrix} \frac{1}{8} \begin{bmatrix} 8 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 40 \\ 80 \end{bmatrix}$

$= \begin{bmatrix} \left(\frac{2}{5}\right)^k & \left(\frac{6}{5}\right)^k \\ 0 & 8\left(\frac{6}{5}\right)^k \end{bmatrix} \begin{bmatrix} 30 \\ 10 \end{bmatrix} = \begin{bmatrix} 30\left(\frac{2}{5}\right)^k + 10\left(\frac{6}{5}\right)^k \\ 80\left(\frac{6}{5}\right)^k \end{bmatrix}$