MAT 2342 – Assignment 1 Solutions

Problem A

Transition matrix $P = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0 & 0.5 & 0.1 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}$

Here: state 1 = subscription to newspaper A
state 2 = subscription to newspaper B
state 3 = subscription to neither

(1) We have the initial state vector $S_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ describing the present state of the Smiths. We are looking for S_2:

$$S_2 = P^2 S_0 = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0 & 0.5 & 0.1 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \underset{\substack{\uparrow \\ \text{on-line} \\ \text{solver}}}{=} \begin{bmatrix} 0.54 \\ 0.26 \\ 0.2 \end{bmatrix}$$

Ans. The probability that the Smiths subscribe to neither paper in 2 years is 0.2.

(2) Now, the initial state vector is $S_0 = \begin{bmatrix} 0.25 \\ 0.1 \\ 0.65 \end{bmatrix}$, where

$0.25 = \frac{50{,}000}{200{,}000}$, $0.1 = \frac{20{,}000}{200{,}000}$, and $0.65 = \frac{13{,}000}{200{,}000}$ are the probabilities that a randomly selected household is in state 1, state 2, and state 3, resp.

$$S_2 = P^2 S_0 = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0 & 0.5 & 0.1 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}^2 \begin{bmatrix} 0.25 \\ 0.1 \\ 0.65 \end{bmatrix} \underset{\substack{\uparrow \\ \text{on-line}}}{=} \begin{bmatrix} 0.445 \\ 0.109 \\ 0.446 \end{bmatrix}$$

$0.109 \cdot 200{,}000 = 21{,}800$

Ans. 2 years from now, 21,800 households will subscribe to newspaper B.

(3) $P^2 = \begin{bmatrix} 0.68 & 0.54 & 0.34 \\ 0.02 & 0.26 & 0.12 \\ 0.3 & 0.2 & 0.54 \end{bmatrix}$

Since P^2 has only strictly positive entries, P is a regular matrix, and this Markov chain possesses a steady-state vector - yes, the numbers will stabilize.

(4) We are looking for the steady-state vector S:

$S = PS$

$(I-P)S = 0$

$$\left[\begin{array}{ccc|c} 0.2 & -0.4 & -0.2 & 0 \\ 0 & 0.5 & -0.1 & 0 \\ -0.2 & -0.1 & 0.3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0.5 & -0.1 & 0 \\ 0 & -0.5 & 0.1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & -0.2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$s_3 = t$

$s_2 - 0.2t = 0 \rightsquigarrow s_2 = 0.2t$

$s_1 - 2s_2 - s_3 = 0 \rightsquigarrow s_1 = 2 \cdot 0.2t + t = 1.4t$

$S = \begin{bmatrix} 1.4t \\ 0.2t \\ t \end{bmatrix}$ general solution

To find a solution that is a probability vector:

$1.4t + 0.2t + t = 1$

$2.6t = 1$

$t = \frac{5}{13}$

$S = \begin{bmatrix} \frac{7}{13} \\ \frac{1}{13} \\ \frac{5}{13} \end{bmatrix}$ steady-state vector

Ans. In the long run $\frac{7}{13} =$ households will subscribe to newspaper A.

Problem B

(1) $c_A(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda+1 & 0 & -1 \\ -5 & \lambda-3 & -2 \\ -6 & 0 & \lambda-4 \end{bmatrix}$

$$= (\lambda+1) \det \begin{bmatrix} \lambda-3 & -2 \\ 0 & \lambda-4 \end{bmatrix} + (-1) \det \begin{bmatrix} -5 & \lambda-3 \\ -6 & 0 \end{bmatrix}$$

$$= (\lambda+1)(\lambda-3)(\lambda-4) - (-1)(-6)(\lambda-3)$$

$$= (\lambda-3)(\lambda^2-3\lambda-4-6) = (\lambda-3)(\lambda^2-3\lambda-10)$$

$$c_A(\lambda) = (\lambda-3)(\lambda+2)(\lambda-5)$$

or $c_A(\lambda) = \lambda^3 - 6\lambda^2 - \lambda + 30$

(2) $\lambda_1 = -2$

$\lambda_2 = 3$

$\lambda_3 = 5$

(3) Eigenvectors for $\lambda_1 = -2$:

$$\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ -5 & -5 & -2 & 0 \\ -6 & 0 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$X_1 = \begin{bmatrix} -t \\ \frac{3}{5}t \\ t \end{bmatrix} = \frac{t}{5} \begin{bmatrix} -5 \\ 3 \\ 5 \end{bmatrix}$ for all $t \neq 0$

Eigenvectors for $\lambda_2 = 3$:

$$\left[\begin{array}{ccc|c} 4 & 0 & -1 & 0 \\ -5 & 0 & -2 & 0 \\ -6 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{13}{4} & 0 \\ 0 & 0 & -\frac{10}{4} & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$X_2 = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{for all } t \neq 0$$

Eigenvectors for $\lambda_3 = 5$:

$$\left[\begin{array}{ccc|c} 6 & 0 & -1 & 0 \\ -5 & 2 & -2 & 0 \\ -6 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 2 & -\frac{17}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{17}{12} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$X_3 = \begin{bmatrix} \frac{1}{6}t \\ \frac{17}{12}t \\ t \end{bmatrix} = \frac{1}{12}t \begin{bmatrix} 2 \\ 17 \\ 12 \end{bmatrix} \quad \text{for all } t \neq 0$$

(4)
$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad P = \begin{bmatrix} -5 & 0 & 2 \\ 3 & 1 & 17 \\ 5 & 0 & 12 \end{bmatrix}$$

(5)

$$A^k = (PDP^{-1})^k = PD^kP^{-1}$$

To find P^{-1}:

$$\left[\begin{array}{ccc|ccc} -5 & 0 & 2 & 1 & 0 & 0 \\ 3 & 1 & 17 & 0 & 1 & 0 \\ 5 & 0 & 12 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{91}{5} & \frac{3}{5} & 1 & 0 \\ 0 & 0 & 14 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{91}{5} & \frac{3}{5} & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{14} & 0 & \frac{1}{14} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{6}{35} & 0 & \frac{1}{35} \\ 0 & 1 & 0 & -\frac{7}{10} & 1 & -\frac{13}{10} \\ 0 & 0 & 1 & \frac{1}{14} & 0 & \frac{1}{14} \end{array}\right]$$

Hence
$$P^{-1} = \begin{bmatrix} -\frac{6}{35} & 0 & \frac{1}{35} \\ -\frac{7}{10} & 1 & -\frac{13}{10} \\ \frac{1}{14} & 0 & \frac{1}{14} \end{bmatrix}$$

Now

$$A^k = \begin{bmatrix} -5 & 0 & 2 \\ 3 & 1 & 17 \\ 5 & 0 & 12 \end{bmatrix} \begin{bmatrix} (-2)^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 5^k \end{bmatrix} \begin{bmatrix} -\frac{6}{35} & 0 & \frac{1}{35} \\ -\frac{7}{10} & 1 & -\frac{13}{10} \\ \frac{1}{14} & 0 & \frac{1}{14} \end{bmatrix}$$

$$= \begin{bmatrix} -5(-2)^k & 0 & 2\cdot 5^k \\ 3(-2)^k & 3^k & 17\cdot 5^k \\ 5(-2)^k & 0 & 12\cdot 5^k \end{bmatrix} \begin{bmatrix} -\frac{6}{35} & 0 & \frac{1}{35} \\ -\frac{7}{10} & 1 & -\frac{13}{10} \\ \frac{1}{14} & 0 & \frac{1}{14} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{7}(-2)^k + \frac{1}{7}\cdot 5^k & 0 & -\frac{1}{7}(-2)^k + \frac{1}{7}5^k \\ -\frac{18}{35}(-2)^k - \frac{7}{10}3^k + \frac{17}{14}5^k & 3^k & \frac{3}{35}(-2)^k - \frac{13}{10}3^k + \frac{17}{14}\cdot 5^k \\ -\frac{6}{7}(-2)^k + \frac{6}{7}\cdot 5^k & 0 & \frac{1}{7}(-2)^k + \frac{6}{7}5^k \end{bmatrix}$$