MAT 1320 Practice Exam Solutions

(1) Find the derivative of $f(x) = \frac{1}{2x+5}$ using the definition. *Do not use differentiation rules.*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{\frac{1}{2(x+h)+5} - \frac{1}{2x+5}}{h}$$

$$= \lim_{h \to 0} \frac{(2x+5) - (2x+2h+5)}{h(2x+5)(2x+2h+5)}$$

$$= \lim_{h \to 0} \frac{-2h}{h(2x+5)(2x+2h+5)}$$

$$= \lim_{h \to 0} \frac{-2}{(2x+5)(2x+2h+5)} = -\frac{2}{(2x+5)^2}$$

(2) Give the derivative of each function below. *No justification is required.*

(a) $f(t) = \arccos(-3t)$

(b) $g(x) = (4\cos x)^5$

(c) $h(u) = \dfrac{3u}{2u+7}$

(a) $f'(t) = -\dfrac{-3}{\sqrt{1-(-3t)^2}} = \dfrac{3}{\sqrt{1-9t^2}}$

(b) $g'(x) = 5(4\cos x)^4 \cdot 4(-\sin x) = -20(4\cos x)^4 \sin x$

(c) $h'(u) = \dfrac{3(2u+7) - 3u \cdot 2}{(2u+7)^2} = \dfrac{21}{(2u+7)^2}$

(3) Using logarithmic differentiation, determine the derivative of $f(x) = (2 + x^5)^{\sin(x)}$. *Your answer should be completely in terms of x.*

$$y = (2+x^5)^{\sin x}$$

$$\ln y = \ln\left((2+x^5)^{\sin x}\right)$$

$$\ln y = \sin x \cdot \ln(2+x^5)$$

$$\frac{y'}{y} = \cos x \cdot \ln(2+x^5) + \sin x \frac{1}{2+x^5}(5x^4)$$

$$y' = y\left(\cos x \ln(2+x^5) + \sin x \frac{5x^4}{2+x^5}\right)$$

$$y' = (2+x^5)^{\sin x}\left(\cos x \cdot \ln(2+x^5) + \sin x \frac{5x^4}{2+x^5}\right)$$

(4) Find the function $g(t)$ satisfying $g'(t) = t^4 + \sin(t)$ and $g(0) = 3$.

$$g(t) = \frac{1}{5}t^5 - \cos(t) + C$$

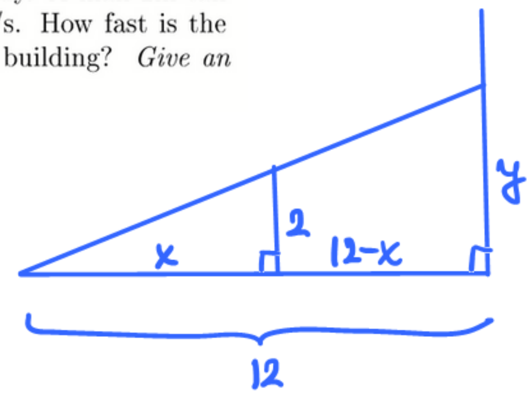
$$g(0) = 0 - \cos(0) + C = 3$$

$$-1 + C = 3$$

$$C = 4$$

$$g(t) = \frac{1}{5}t^5 - \cos(t) + 4$$

(5) A spotlight on the ground shines towards a wall of a building 12m away. A man 2m tall walks away from the spotlight towards the building at a speed 1.6m/s. How fast is the length of his shadow on the wall decreasing when he is 4m from the building? *Give an exact, verbal answer. Draw a diagram!*



Given: $\frac{dx}{dt} = 1.6 m/s$

Want: $\left.\frac{dy}{dt}\right|_{x=8} = ?$

Similar triangles: $\frac{y}{12} = \frac{2}{x}$

$$y = 24x^{-1}$$

$$\frac{dy}{dt} = -24x^{-2}\frac{dx}{dt}$$

$$\left.\frac{dy}{dt}\right|_{x=8} = -\frac{24}{8^2}1.6 = -\frac{24\cdot 16}{8^2\cdot 10} = -\frac{3}{5}$$

Ans: The length of the man's shadow on the wall is decreasing at a rate 0.6m/s.

(6) Consider the curve described by $x^4 + xy^5 + y^2 = 4$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

(b) Find the y-coordinate of each point on the curve with $x = 0$. Then find the slope of the tangent line to the curve at each such point.

(a) $4x^3 + y^5 + x \cdot 5y^4 \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$

$(5xy^4 + 2y) \frac{dy}{dx} = -(4x^3 + y^5)$

$$\frac{dy}{dx} = -\frac{4x^3 + y^5}{5xy^4 + 2y}$$

(b) $x^4 + xy^5 + y^2 = 4$ and $x = 0 \Rightarrow y^2 = 4$

$\Rightarrow y = \pm 2$

$\left.\frac{dy}{dx}\right|_{x=0} = -\frac{1}{2} y^4$

Point $(0,2)$: $\frac{dy}{dx} = -\frac{1}{2} \cdot 2^4 = -8$

Point $(0,-2)$: $\frac{dy}{dx} = -\frac{1}{2} \cdot (-2)^4 = -8$

(7) Evaluate

$$\frac{d}{dx}\int_x^{3x^2} \frac{\sin(t)}{\ln(t+4)}\, dt. =$$

Let $f(t) = \frac{\sin(t)}{\ln(t+4)}$.

using FTC I :

$$= \frac{d}{dx}\int_x^0 f(t)dt + \frac{d}{dx}\int_0^{3x^2} f(t)\,dt =$$

$$= -\frac{d}{dx}\int_0^x f(t)dt + \frac{d}{dx}\int_0^{3x^2} f(t)dt$$

Let
$u = 3x^2$
$\frac{du}{dx} = 6x$

$$= -f(x) + \frac{d}{du}\int_0^u f(t)dt \cdot \frac{du}{dx}$$

$$= -f(x) + f(u)\cdot 6x = -f(x) + 6x\, f(3x^2)$$

$$= -\frac{\sin x}{\ln(x+4)} + 6x\,\frac{\sin(3x^2)}{\ln(3x^2+4)}$$

(8) For each function $f(x)$ below, give an antiderivative. You do not need to include $+C$. *No justification is required.*

↑ F(x)

(a) $f(x) = x^3 + \frac{1}{\sqrt{x}}$

(b) $f(x) = \sec(2x)\tan(2x)$

(c) $f(x) = \frac{3}{x+4}$

(a) $F(x) = \frac{1}{4}x^4 + 2\sqrt{x}$

(b) $F(x) = \frac{1}{2}\sec(2x)$

(c) $F(x) = 3\ln|x+4|$

(9) Evaluate each of the following integrals. *Show your work!*

(a) $\int_2^6 e^{-x}\sin(e^{-x})\,dx$

(b) $\int \cos(\ln t)\,dt$

(a) $\int_2^6 e^{-x}\sin(e^{-x})dx = -\int_{e^{-2}}^{e^{-6}} \sin(t)dt$

Sub. $t = e^{-x}$
$dt = -e^{-x}dx$

$= \cos t \Big|_{e^{-2}}^{e^{-6}} = \cos(e^{-6}) - \cos(e^{-2})$

(b) $\int \cos(\ln t)dt =$

$= \int \cos(x)e^{x}dx$

Subst.
$x = \ln t$
$dx = \frac{1}{t}dt$
$t = e^{x}$
$dt = e^{x}dx$

Let $I = \int \cos(x)e^{x}dx$

$= e^{x}\sin x - \int e^{x}\sin x\,dx$

By Parts:
$u = e^{x}$ $\quad dv = \cos x\,dx$
$du = e^{x}dx$ $\quad v = \sin x$

$= e^{x}\sin x - \left(-e^{x}\cos x + \int e^{x}\cos x\,dx\right)$

By Parts:
$u = e^{x}$ $\quad dv = \sin x\,dx$
$du = e^{x}dx$ $\quad v = -\cos x$

$= e^{x}(\sin x + \cos x) - I$

So $2I = e^{x}(\sin x + \cos x)$

$I = \frac{1}{2}e^{x}(\sin x + \cos x) + C$

$\int \cos(\ln t)dt = \frac{1}{2}t\left(\sin(\ln t) + \cos(\ln t)\right)$

(10) Using the method of partial fractions, evaluate $\int \frac{4x^2+2}{x(x^2+2)}\,dx$. *Give your final answer in terms of the original variable.*

$$\frac{4x^2+2}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$

$$4x^2+2 = A(x^2+2) + (Bx+C)x$$

$$4x^2+2 = (A+B)x^2 + Cx + 2A$$

$$\left.\begin{array}{l} 2A = 2 \\ C = 0 \\ A+B = 4 \end{array}\right\} \Rightarrow \begin{array}{l} A = 1 \\ B = 3 \\ C = 0 \end{array}$$

$$\int \frac{4x^2+2}{x(x^2+1)}dx = \int\left(\frac{1}{x} + \frac{3x}{x^2+2}\right)dx =$$

$$= \int \frac{dx}{x} - \frac{3}{2}\int \frac{2x}{x^2+2}\,dx$$

$$= \ln|x| - \frac{3}{2}\ln|x^2+2| + C$$

(11) Using the method of trigonometric substitution, evaluate $\int \sqrt{1-4x^2}\,dx$. *Give your final answer in terms of the original variable.*

$$\int \sqrt{1-4x^2}\,dx$$

$$= \frac{1}{2}\int \cos\varphi \cos\varphi\, d\varphi$$

$$= \frac{1}{2}\int \cos^2\varphi\, d\varphi$$

$$= \frac{1}{4}\int (1+\cos 2\varphi)\,d\varphi$$

$$= \frac{1}{4}\left(\varphi + \frac{1}{2}\sin 2\varphi\right) + C$$

$$= \frac{1}{4}\left(\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\right) + C$$

subs. $2x = \sin\varphi$

$2\,dx = \cos\varphi\, d\varphi$

$\sqrt{1-4x^2} = \cos\varphi$

$\sin 2\varphi =$

$= 2\sin\varphi\cos\varphi$

$= 4x\sqrt{1-4x^2}$

(12) We wish to find a numerical approximation for the integral $\int_0^2 e^{x^2}\,dx$.

(a) Give the expression for the Riemann sum using right endpoints with $n = 4$.

(b) Give the expression for the approximation using Simpson's method with $n = 4$.

(c) Determine the smallest value of n that will guarantee that the approximation error using the trapezoid method for the integral $\int_0^2 e^{x^2}\,dx$ is at most 0.002.

$f(x) = e^{x^2}$, $a=0$, $b=2$, $n=4$, $\Delta x = \frac{b-a}{n} = \frac{1}{2}$, $x_i = \frac{i}{2}$

(a) $R_4 = \sum_{i=1}^{4} f(x_i)\Delta x$

$= \frac{1}{2}\left(f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + f(2)\right)$

$= \frac{1}{2}\left(e^{\frac{1}{4}} + e + e^{\frac{9}{4}} + e^4 \right)$

(b) $S_4 = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$

$= \frac{1}{12}\left(f(0) + 4f(\tfrac{1}{2}) + 2f(1) + 4f(\tfrac{3}{2}) + f(2)\right)$

$= \frac{1}{12}\left(1 + 4e^{\frac{1}{4}} + 2e + 4e^{\frac{9}{4}} + e^4 \right)$

(c) $|E_{T_n}| \leq \frac{K(b-a)^3}{12n^2}$ if $|f''(x)| \leq K$ on $[a,b] = [0,2]$

$|E_{T_n}| \leq \frac{K \cdot 8}{12n^2}$

$f''(x) = (4x^2+2)e^{x^2}$

$|f''(x)| = |(4x^2+2)e^{x^2}| \leq$

$\leq |4x^2+2|\,|e^{x^2}|$

$\leq 18 \cdot e^4$ on $[0,2]$

Let $K = 18e^4$

It suffices that $\frac{8K}{12n^2} = \frac{144e^4}{12n^2} = \frac{12e^4}{n^2} \leq 0.002$

so $n^2 \geq \frac{12e^4}{0.002}$

$n \geq \sqrt{6000e^4} = 10e^2\sqrt{60} \doteq 572.4$

Ans: $n \geq \underline{573}$

(13) Find the absolute maximum and absolute minimum values, as well as the locations (x-coordinate) where they occur, for the function $f(x) = \frac{\ln x}{x^3}$ on the closed interval $[\frac{1}{e}, e]$.

$$f'(x) = \frac{\frac{1}{x} \cdot x^3 - \ln x \cdot 3x^2}{x^6}$$

$$= \frac{1 - 3\ln x}{x^4}$$

$f'(x)$ exists for all $x \in [\frac{1}{e}, e]$

$f'(x) = 0 \Rightarrow 1 - 3\ln x = 0 \Rightarrow \ln x = \frac{1}{3} \Rightarrow x = e^{\frac{1}{3}}$

Critical pt: $x = e^{\frac{1}{3}}$, $f(x) = \frac{\ln e^{\frac{1}{3}}}{(e^{\frac{1}{3}})^3} = \frac{1}{3}e \doteq 0.91$

Endpts:

$x = \frac{1}{e}$, $f(x) = \frac{\ln(\frac{1}{e})}{(\frac{1}{e})^3} = \frac{-1}{\frac{1}{e^3}} = -e^3 \doteq -20.09$

$x = e$, $f(x) = \frac{\ln(e)}{e^3} = \frac{1}{e^3} \doteq 0.05$

$\Rightarrow$ Abs. max: $(e^{\frac{1}{3}}, \frac{1}{3}e)$

$\Rightarrow$ Abs. min: $(\frac{1}{e}, -e^3)$

(14) Determine each of the following limits.

(a) $\lim_{t \to \infty} \frac{t^2 + 5t}{e^t + 2}$

(b) $\lim_{x \to 0} (1 - x)^{\frac{1}{x}}$

(a) $$\lim_{t \to \infty} \frac{t^2 + 5t}{e^t + 2} \overset{L'H}{=} \lim_{t \to \infty} \frac{2t + 5}{e^t}$$

$$\overset{L'H}{=} \lim_{t \to \infty} \frac{2}{e^t}$$

$$= 0$$

(b)

$$y = (1-x)^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln(1-x)$$

$$\lim_{x\to 0}(\ln y) = \lim_{x\to 0}\frac{\ln(1-x)}{x} =$$

$$\overset{L'H}{=} \lim_{x\to 0}\frac{-1}{1-x} = -1$$

$$\lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{\lim_{x\to 0}(\ln y)} = e^{-1}$$

(15) Consider the function $f(x) = \frac{2(x^2-1)}{x^3}$.

(a) Give the equations of all horizontal and vertical asymptotes. Evaluate the corresponding limits.

(b) Determine the intervals where f is increasing/decreasing. Identity all local maxima and minima.

(c) Determine the intervals where f is concave up/concave down. Identify all inflection points.

(d) Sketch the graph of $f(x) = \frac{2(x^2-1)}{x^3}$, indicating the special features that you found in parts (a) – (c).

(a)

$$\lim_{x\to -\infty}\frac{2(x^2-1)}{x^3} = 2\lim_{x\to -\infty}\frac{1-\frac{1}{x^2}}{x} = 0$$

$$\lim_{x\to \infty}\frac{2(x^2-1)}{x^3} = 2\lim_{x\to \infty}\frac{1-\frac{1}{x^2}}{x} = 0$$

Hence horizontal asymptote $y=0$

Vertical asymptote $x=0$

$$\lim_{x\to 0^-}\frac{2(x^2-1)}{x^3} = \infty \qquad \lim_{x\to 0^+}\frac{2(x^2-1)}{x^3} = -\infty$$

(b)

$$f'(x) = \frac{-2(x^2-3)}{x^4} = \frac{-2(x-\sqrt{3})(x+\sqrt{3})}{x^4}$$

$-\sqrt{3}$ 0 $\sqrt{3}$

$x-\sqrt{3}$	−	−	−	+
$x+\sqrt{3}$	−	+	+	+
$-\frac{2}{x^4}$	−	−	−	−
$f'(x)$	−	+	+	−
$f(x)$	↘	↗	↗	↘

* f is increasing on $(-\sqrt{3},0)$, $(0,\sqrt{3})$
* f is decreasing on $(-\infty,-\sqrt{3})$, $(\sqrt{3},\infty)$
* Extrema: $x=-\sqrt{3}$ $f(x) = -\frac{4}{3\sqrt{3}}$ loc. min
  $x=\sqrt{3}$ $f(x) = \frac{4}{3\sqrt{3}}$ loc. max

(c)

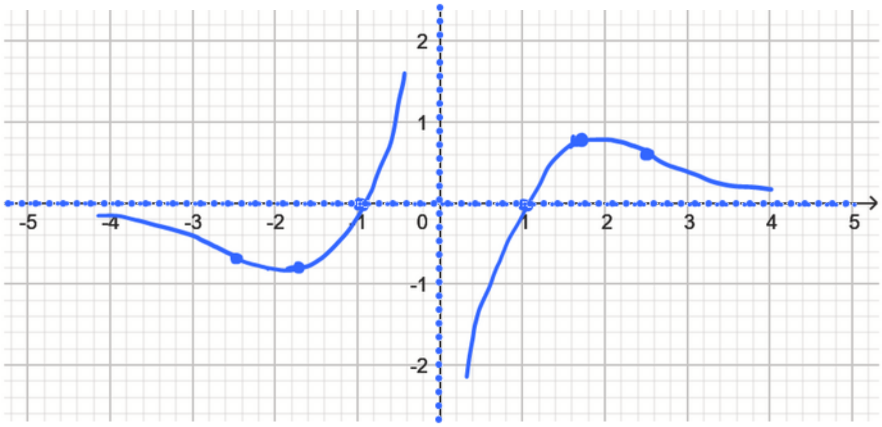
$$f''(x) = \frac{4(x^2-6)}{x^5} = \frac{4(x-\sqrt{6})(x+\sqrt{6})}{x^5}$$

$-\sqrt{6}$ 0 $\sqrt{6}$

$x-\sqrt{6}$	−	−	−	+
$x+\sqrt{6}$	−	+	+	+
$\frac{4}{x^5}$	−	−	+	+
$f''(x)$	−	+	−	+
$f(x)$	∩	∪	∩	∪

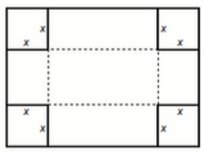
* f is concave up on $(-\sqrt{6},0)$, $(\sqrt{6},\infty)$
* f is concave down on $(-\infty,-\sqrt{6})$, $(0,\sqrt{6})$
* inflection points: $\left(-\sqrt{6}, -\frac{5}{3\sqrt{6}}\right)$, $\left(\sqrt{6}, \frac{5}{3\sqrt{6}}\right)$

(d)



(16) Suppose you have a rectangular piece of cardboard measuring 2ft × 3ft. You would like to make a box with an open top by cutting out a square of side length x from each corner (where x is to be determined), and then fold up the sides. (See the figure.)

Determine x (in ft) so that your box has the largest possible volume.



$$V = x(2-2x)(3-2x)$$

$$= 4x^3 - 10x^2 + 6x$$

$$V' = 12x^2 - 20x + 6$$

$$= 12(x - x_1)(x - x_2)$$

Find abs. max on $[0,1]$

$$x_{1,2} = \frac{20 \pm \sqrt{400 - 24 \cdot 12}}{24} = \frac{20 \pm 4\sqrt{7}}{24} = \frac{5 \pm \sqrt{7}}{6}$$

$$x_1 = \frac{5-\sqrt{7}}{6} \doteq 0.4$$

$$x_2 = \frac{5+\sqrt{7}}{6} \doteq 1.3$$

		x_1		x_2	
$x - x_1$	$-$		$+$		$+$
$x - x_2$	$-$		$-$		$+$
V'	$+$		$-$		$+$
V	↗		↘		↗

↑ local & abs. max (at x_1)

Ans: The volume is max for $x = \frac{5-\sqrt{7}}{6} \doteq 0.4$ ft