MULTIPLE-CHOICE QUESTIONS Questions 1–10 are multiple choice format worth 2 points each. Answers to multiple-choice questions do not need to be justified. You may write your scrap work on your paper but it will not be graded. When you reach your answer, clearly indicate the question number and write the letter of your response beside the question number: For example: (*write out your scrap work, but it will not be graded)*

(*clearly indicate your final choice*) **Q1.** [`letter of your choice`]

Q1. Which one of the following statements is always true? Solution: **D**

A. If a function is continuous at a, then it is differentiable at a.

B. If a function is continuous, then is has an absolute maximum.

C. If f attains its global maximum on a closed interval at a point p, then $f'(p) = 0$.

D. If a function is differentiable at a, then it is continuous at a.

E. If $f''(p) = 0$, then f has an inflection point at p.

F. If p is a critical number of f, then f has a local minimum or a local maximum at p.

Q2. For the function $f(x) = xe^{3-(x/4)}$, which one of the following statements is true? Solution: **B**

A. $x = 4$ is a critical number at which f has a local minimum.

B. $x = 4$ is a critical number at which f has a local maximum.

C. $x = 4$ is a critical number at which f has an inflection point.

D. $x = 4$ is not a critical number but f has an inflection point there.

E. $x = 4$ is a critical number but f does not have a local extremum nor an inflection point there.

F. $x = 4$ is not in the domain of f.

Q3. For which value of the parameter a is the following function **continuous**?

$$f(x) = \begin{cases} \dfrac{x^2 - 4x - 32}{x - 8} & \text{if } x \neq 8 \\ a & \text{if } x = 8 \end{cases}$$

Solution: **H**

A. -4 **B.** 1 **C.** 10 **D.** 9 **E.** 2 **F.** 0 **G.** 8 **H.** 12

I. There is no such value of a.

Q4. The limit below represents the derivative $f'(a)$ for some function $f(x)$ at some number a.

$$\lim_{h\to 0} \frac{\sqrt{\frac{1}{4}+h}-\frac{1}{2}}{h}$$

For which of the following functions does the above limit define $f'(a)$ for the given number a ?

Solution: **E**

A. $f'(\frac{1}{2})$ for $f(x)=\sqrt{x}$ **B.** $f'(\frac{1}{16})$ for $f(x)=\sqrt{x}$ **C.** $f'(4)$ for $f(x)=\frac{1}{\sqrt{x}}$

D. $f'(2)$ for $f(x)=\frac{1}{\sqrt{x}}$ **E.** $f'(\frac{1}{4})$ for $f(x)=\sqrt{x}$ **F.** $f'(16)$ for $f(x)=\frac{1}{\sqrt{x}}$

Q5. Consider the curve defined implicitly by the equation

$$\sin(x^2+y)=3x+3y^3.$$

Find the slope of the tangent line to this curve at the point $(1,-1)$.

Solution: **C** **A.** $-\frac{3}{14}$ **B.** $-\frac{2}{19}$ **C.** $-\frac{1}{8}$ **D.** $-\frac{2}{11}$ **E.** $\frac{1}{4}$ **F.** 0 **G.** $-\frac{1}{14}$ **H.** $\frac{1}{2}$

Q6. Find the linearization of $f(x)=\arctan(1-3x)$ at 0.

Solution: **G** **A.** $L(x)=\frac{\pi}{4}-2x$ **B.** $L(x)=\frac{\pi}{4}x-2$ **C.** $L(x)=\frac{\pi}{4}x-1$ **D.** $L(x)=\frac{\pi}{4}x-\frac{3}{2}$

E. $L(x)=\frac{\pi}{4}-\frac{1}{2}x$ **F.** $L(x)=\frac{\pi}{4}x-\frac{1}{2}$ **G.** $L(x)=\frac{\pi}{4}-\frac{3}{2}x$ **H.** $L(x)=\frac{\pi}{4}-x$

Q7. Suppose you know $\int_1^2 f(x)\,dx=-1.$ and $\int_7^2 f(x)\,dx=2.$ Find $\int_1^7 f(x)\,dx.$

Solution: **A** **A.** -3 **B.** 3 **C.** -2 **D.** 2 **E.** -1 **F.** 1 **G.** 0

Q8. A spherical snowball is melting in the sun. At all times, the surface area A of this snowball is related to its diameter D by the equation $A=\pi D^2$.

Find the rate of change in the snowball's diameter at the moment in time when A is decreasing at a rate of 3 cm^2/s and $D=\frac{7}{\pi}$ cm. Solution: **B**

A. D is decreasing at a rate of $\frac{1}{4}$ cm/s

B. D is decreasing at a rate of $\frac{3}{14}$ cm/s

C. D is decreasing at a rate of $\frac{5}{16}$ cm/s

D. D is decreasing at a rate of $\frac{7}{8}$ cm/s

E. D is increasing at a rate of $\frac{1}{4}$ cm/s

F. D is increasing at a rate of $\frac{3}{14}$ cm/s

G. D is increasing at a rate of $\frac{5}{16}$ cm/s

H. D is increasing at a rate of $\frac{7}{8}$ cm/s

Q9. Find $f(x)$ given $f'(x) = x^5 \ln(x)$ and $f(1) = 0$

Solution: **E** **A.** $f(x) = 5x^4 \ln(x) + x^4 + C$ **B.** $f(x) = \frac{1}{6}x^6 \ln(x) - \frac{1}{36}x^6$ **C.** $f(x) = 5x^4 \ln(x) + x^4$

D. $f(x) = \frac{1}{5}x^5 \ln(x) - \frac{1}{25}x^5 + \frac{1}{25}$ **E.** $f(x) = \frac{1}{6}x^6 \ln(x) - \frac{1}{36}x^6 + \frac{1}{36}$ **F.** $f(x) = 6x^5 \ln(x) + x^5$

G. $f(x) = 6x^5 \ln(x) + x^5 + C$ **H.** $f(x) = \frac{1}{5}x^5 \ln(x) - \frac{1}{25}$ **I.** $f(x) = \frac{1}{6}x^6(x \ln(x) - x) + C$

Q10. Find the absolute maximum and minimum values of the function $f(x) = (x^2 - 3x + 5)e^{-x/3}$ on the closed interval $[0, 10]$.

Solution: **F**

A. absolute max. value 3.200... absolute min. value 1.540...
B. absolute max. value 5.250... absolute min. value 1.997...
C. absolute max. value 5.250... absolute min. value 2.675...
D. absolute max. value 5 absolute min. value 2.675...
E. absolute max. value 5.250... absolute min. value 3.200...
F. absolute max. value 5 absolute min. value 1.540...

LONG-ANSWER QUESTIONS

For long-answer questions, all of your work must be justified and your steps must be written in a clear and logical order, using correct mathematical notation. Clearly indicate Question numbers.

For example: **Q11 a).** [write a fully justified solution].

Q11. Evaluate each of the following limits. For each you must show ALL your steps and use appropriate calculus and algebraic methods seen in class. If a limit does not exist, you must justify how you reached this conclusion. Identify the types of any indeterminate forms you encounter along the way.

a) $\lim_{x \to 0} (\cos x)^{1/x^2}$ b) $\lim_{t \to \infty} \frac{e^t + 1}{t^2 + t}$ c) $\lim_{x \to \infty} \frac{x^2 + \sqrt{x^6 + x^4}}{2x - 4x^3}$

Solution: a)

indet. form 1^∞

$$\lim_{x\to 0}(\cos x)^{1/x^2} = \lim_{x\to 0} e^{\ln((\cos x)^{1/x^2})} = \lim_{x\to 0} e^{\frac{1}{x^2}\ln((\cos x))}$$

Consider

$$\lim_{x\to 0}\frac{1}{x^2}\ln((\cos x)) = \lim_{x\to 0}\frac{\ln((\cos x))}{x^2} \qquad \text{type } \tfrac{0}{0}$$

$$= \lim_{x\to 0}\frac{\frac{-\sin x}{\cos x}}{2x} \qquad \text{l'Hospital's rule}$$

$$= \lim_{x\to 0}\frac{-\tan(x)}{2x} \qquad \text{type } \tfrac{0}{0}$$

$$= \lim_{x\to 0}\frac{-\sec^2(x)}{2} \qquad \text{l'Hospital's rule}$$

$$= -\frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0}(\cos x)^{1/x^2} = \lim_{x\to 0} e^{\frac{1}{x^2}\ln((\cos x))} = e^{-1/2}$$

since e^x is continous at $x = -1/2$.

Solution: b)

indet. form $\frac{\infty}{\infty}$

$$\lim_{t\to\infty}\frac{e^t+1}{t^2+t} = \lim_{t\to\infty}\frac{e^t}{2t+1} \qquad \text{l'Hosp., now indet. form } \tfrac{\infty}{\infty}$$

$$= \lim_{t\to\infty}\frac{e^t}{2} \qquad \text{l'Hosp.}$$

$$= \infty$$

Since the limit does not approach a unique real number, this limit does not exist.

Solution: b)

indet. form $\frac{\infty}{\infty}$

$$\begin{aligned}
\lim_{x\to\infty}\frac{x^2+\sqrt{x^6+x^4}}{2x-4x^3} &= \lim_{x\to\infty}\frac{x^2+\sqrt{x^6(1+\frac{1}{x^2})}}{x^3(\frac{2}{x^2}-4)}\\
&= \lim_{x\to\infty}\frac{x^2+|x^3|\sqrt{1+\frac{1}{x^2}}}{x^3(\frac{2}{x^2}-4)}\\
&= \lim_{x\to\infty}\frac{x^2+x^3\sqrt{1+\frac{1}{x^2}}}{x^3(\frac{2}{x^2}-4)} \qquad \text{since } x^3>0 \text{ as } x\to\infty\\
&= \lim_{x\to\infty}\frac{x^3(\frac{1}{x}+\sqrt{1+\frac{1}{x^2}})}{x^3(\frac{2}{x^2}-4)}\\
&= \lim_{x\to\infty}\frac{(\frac{1}{x}+\sqrt{1+\frac{1}{x^2}})}{(\frac{2}{x^2}-4)}\\
&= \frac{0+\sqrt{1+0}}{0-4}\\
&= -\frac{1}{4}
\end{aligned}$$

Q12. Use logarithmic differentiation to determine the derivative of $g(x)=(x^2+1)^{\cos(x)}$. Show all your work and write your final answer completely in terms of x.

Solution:

$$\begin{aligned}
y &= (x^2+1)^{\cos(x)}\\
\Rightarrow \ln|y| &= \ln|(x^2+1)^{\cos(x)}|\\
\Rightarrow \ln|y| &= \cos(x)\ln|x^2+1| = \cos(x)\ln(x^2+1) \qquad \text{since } x^2+1>0\\
\Rightarrow \frac{1}{y}y' &= -\sin(x)\ln(x^2+1)+\cos(x)\frac{2x}{x^2+1}\\
\Rightarrow y' &= y\left(-\sin(x)\ln(x^2+1)+\cos(x)\frac{2x}{x^2+1}\right)\\
\Rightarrow y' &= (x^2+1)^{\cos(x)}\left(-\sin(x)\ln(x^2+1)+\cos(x)\frac{2x}{x^2+1}\right)
\end{aligned}$$

Q13. Evaluate each of the following integrals. Show ALL your work for each one.

a) $\displaystyle\int \frac{1}{x(x^2+1)}\,dx$ b) $\displaystyle\int_2^3 e^t\cos(e^t)\,dt$ c) $\displaystyle\int_1^4 e^{\sqrt{x}}\,dx$

Solution: a) Using the method of partial fractions, we will find A, B, C such that

$$\frac{1}{x(1+x^2)}=\frac{A}{x}+\frac{Bx+C}{1+x^2}=\frac{A(1+x^2)+(Bx+C)x}{x(1+x^2)}$$

$$\Rightarrow 1 = A(1+x^2) + (Bx+C)x = A + Ax^2 + Bx^2 + Cx = (A+B)x^2 + Cx + A$$

Comparing coefficients, the left side is $0x^2 + 0x + 1$ and the right side is $(A+B)x^2 + Cx + A$.

Thus, $0 = A + B$, $0 = C$ and $A = 1$, whence $B = -A = -1$.

Thus, we obtain:

$$\begin{aligned}
\int \frac{1}{x(x^2+1)}\,dx &= \int \left(\frac{1}{x} + \frac{-x+0}{1+x^2}\right)dx \\
&= \ln|x| - \int \frac{x}{1+x^2}\,dx && u = 1+x^2 \Rightarrow \tfrac{du}{dx} = 2x \\
&= \ln|x| - \int \frac{x}{u}\frac{du}{2x} \\
&= \ln|x| - \frac{1}{2}\int \frac{1}{u}\,du \\
&= \ln|x| - \frac{1}{2}\ln|u| + C && \text{do not forget the } +C\ ! \\
&= \ln|x| - \frac{1}{2}\ln|1+x^2| + C
\end{aligned}$$

Solution: b)

sub. $u = e^t \Rightarrow \frac{du}{dt} = e^t \Rightarrow dt = \frac{du}{e^t}$

$$\begin{aligned}
\int_2^3 e^t\cos(e^t)\,dt &= \int_{u=e^2}^{u=e^3} u\cos(u)\frac{du}{e^t} && t = 3 \Rightarrow u = e^3 \\
&= \int_{e^2}^{e^3} u\cos(u)\frac{du}{u} && t = 2 \Rightarrow u = e^2 \\
&= \int_{e^2}^{e^3} \cos(u)\,du \\
&= [\sin(u)]_{e^2}^{e^3} \\
&= \sin(e^2) - \sin(e^3)
\end{aligned}$$

Solution: c)

sub. $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow \frac{dx}{=} 2\sqrt{x}\,du$

$$\begin{aligned}
&\int_1^4 e^{\sqrt{x}}\,dx && x = 4 \Rightarrow u = \sqrt{4} = 2 \\
&= \int_{\sqrt{1}}^{\sqrt{4}} e^u(2\sqrt{x}\,du) && x = 1 \Rightarrow u = \sqrt{1} = 1 \\
&= \int_1^2 e^u(2u)\,du && \text{parts: } f = 2u \quad g' = e^u \\
&= \left[2ue^u - \int 2e^u\right]_1^2 && f' = 2 \quad g = e^u \\
&= [2ue^u - 2e^u]_1^2 \\
&= \left(2(2)e^2 - 2e^2\right) - \left(2(1)e^1 - 2e^1\right) \\
&= 4e^2 - 2e^2 - 2e + 2e \\
&= 2e^2
\end{aligned}$$

Q14. Consider the integrable piecewise function $f(x)$ given below:

$$f(x) = \begin{cases} -4e^x & \text{if } x < 0 \\ x\sin(x) & \text{if } 0 \le x < \pi \\ 2x^{-1} & \text{if } x \ge \pi \end{cases}$$

Evaluate the definite integral $\displaystyle\int_{-2}^{5} f(x)\,dx$

Show all your work and briefly explain your process. Give the exact final answer.

Solution: First, we can split up the definite integral as

$$\int_{-2}^{5} f(x)\,dx = \int_{-2}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx + \int_{\pi}^{5} f(x)\,dx.$$

We will evaluate each of these separately, then add our answers.

$$\begin{aligned}\int_{-2}^{0} f(x)\,dx &= \int_{-2}^{0} -4e^x\,dx \\ &= [-4e^x]_{-2}^{0} \\ &= -4e^0 - -4e^{-2} \\ &= -4 - (-4)e^{-2}\end{aligned}$$

$$\begin{aligned}\int_{0}^{\pi} f(x)\,dx &= \int_{0}^{\pi} x\sin(x)\,dx && \text{using parts } u = x \text{ and } v' = \sin(x) \\ &= \left[-x\cos(x) - \int (1)(-\cos(x))\,dx\right]_0^{\pi} && u' = 1 \text{ and } v = -\cos(x) \\ &= \left[-x\cos(x) + \int \cos(x)\,dx\right]_0^{\pi} \\ &= [-x\cos(x) + \sin(x)]_0^{\pi} \\ &= [-\pi\cos(\pi) + \sin(\pi)] - [-(0)\cos(0) + \sin(0)] \\ &= -\pi(-1) + 0 - 0 \\ &= \pi\end{aligned}$$

$$\begin{aligned}\int_{\pi}^{5} f(x)\,dx &= \int_{\pi}^{5} 2x^{-1}\,dx \\ &= [2\ln|x|]_{\pi}^{5} \\ &= 2\ln|5| - 2\ln|\pi|\end{aligned}$$

Altogether, we have:

$$\int_{-2}^{5} f(x)\,dx = -4 - (-4)e^{-2} + \pi + 2\ln|5| - 2\ln|\pi|.$$

Q15. (i). By choosing an appropriate trig substitution and simplifying the result, show that

$$\int \frac{1}{(x^2-6x+10)^{5/2}}\,dx = \int \cos^3(\theta)\,d\theta.$$

Solution: We complete the square: $x^2-6x+10 = x^2-6x+9+1 = (x-3)^2+1$.

Now,

$$\begin{aligned}
\int \frac{1}{(x^2-6x+10)^{5/2}}\,dx &= \int \frac{1}{((x-3)^2+1)^{5/2}}\,dx && \text{use trig sub: } x-3=\tan\theta \\
&= \int \frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^{5/2}} && x=\tan\theta+3 \Rightarrow dx=\sec^2\theta\,d\theta \\
&= \int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^{5/2}} && \text{since } \tan^2\theta+1=\sec^2\theta \\
&= \int \frac{\sec^2\theta\,d\theta}{\sec^5\theta} \\
&= \int \frac{1}{\sec^3\theta}\,d\theta \\
&= \int \cos^3\theta\,d\theta
\end{aligned}$$

(ii). Evaluate the integral $\int \cos^3(\theta)\,d\theta$ in terms of θ.

Show ALL your work and use appropriate methods and notation.

Solution:

$$\begin{aligned}
\int \cos^3\theta\,d\theta &= \int \cos\theta\cos^2\theta\,d\theta \\
&= \int \cos\theta(1-\sin^2\theta)\,d\theta && \text{use } u=\sin\theta \Rightarrow du=\cos\theta\,d\theta \\
&= \int (1-u^2)\,du \\
&= u-\frac{1}{3}u^3+C \\
&= \sin\theta-\frac{1}{3}\sin^3\theta+C
\end{aligned}$$

(iii). According to the trig substitution you applied in part (i), express your answer from (ii) in terms of x. Your final answer should not include any trigonometric or inverse trigonometric functions.

Solution: Since $x-3=\tan\theta$, we have $\sin\theta = \dfrac{x-3}{\sqrt{(x-3)^2+1}}$

Thus,

$$\sin\theta-\frac{1}{3}\sin^3\theta+C = \frac{x-3}{\sqrt{(x-3)^2+1}} - \frac{1}{3}\left(\frac{x-3}{\sqrt{(x-3)^2+1}}\right)^3 + C$$

Q16. Consider the following function:

$$f(x) = \frac{x^3}{x^2 - 1}$$

a) Identify all horizontal and vertical asymptotes of f (if any) based on appropriate limits.

Solution: We have $f(x) = \frac{x^3}{(x+1)(x-1)}$ which is undefined when $x = \pm 1$.

$$\lim_{x \to -\infty} \frac{x^3}{x^2 - 1} = \lim_{x \to -\infty} \frac{x^2(x)}{x^2(1 - \frac{1}{x^2})} = \lim_{x \to -\infty} \frac{x}{1 - \frac{1}{x^2}} = \lim_{x \to -\infty} \frac{x}{1 - 0} = -\infty$$

$$\lim_{x \to \infty} \frac{x^3}{x^2 - 1} = \lim_{x \to \infty} \frac{x^2(x)}{x^2(1 - \frac{1}{x^2})} = \lim_{x \to \infty} \frac{x}{1 - \frac{1}{x^2}} = \lim_{x \to \infty} \frac{x}{1 - 0} = \infty$$

$$\lim_{x \to -1^-} \frac{x^3}{x^2 - 1} \to \frac{-1}{0^+} \to -\infty \qquad \lim_{x \to -1^+} \frac{x^3}{x^2 - 1} \to \frac{-1}{0^-} \to \infty$$

$$\lim_{x \to 1^-} \frac{x^3}{x^2 - 1} \to \frac{1}{0^-} \to -\infty \qquad \lim_{x \to 1^+} \frac{x^3}{x^2 - 1} \to \frac{1}{0^+} \to \infty$$

So there are skewed vertical asymptotes at $x = \pm 1$ and there are no horizontal asymptotes.

b) Find and identify the intervals where f is increasing/decreasing. Identify all local maxima and minima. Show ALL your work!

Solution: We find the critical numbers: $f'(x) = \frac{3x^2(x^2-1) - x^3(2x)}{(x^2-1)^2} = \frac{x^4 - 3x^2}{(x^2-1)^2}$.

$0 = f'(x) \Rightarrow 0 = \frac{x^2(x^2-3)}{(x^2-1)^2} \Rightarrow 0 = x, x = \pm\sqrt{3}$.

Now we analyze the relevant intervals of the domain:

interval	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$(\sqrt{3}, \infty)$
sign of f'	+	−	−	−	−	+
behaviour of f	inc ↗	dec ↘	dec ↘	dec ↘	dec ↘	inc ↗

Thus, f has a local max at $x = -\sqrt{3}$ and a local min at $x = \sqrt{3}$.

c) The second derivative of f is given below:

$$f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

Find and identify the intervals where f is concave up/down. Identify all inflection points. Show ALL your work!

Solution: We find IP candidates by solving $0 = f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}$. So the only solution is $x = 0$.

Now we analyze the relevant intervals of the domain:

interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
sign of f''	−	+	−	+
behaviour of f	C.D. ∩	C.U. ∪	C.D. ∩	C.U. ∪

Thus, f has an inflection point at $x = 0$.

d) Sketch the graph of f and clearly identify all special points and features you found in parts a)–c).