Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Calculus I MAT1320

First Midterm Exam (γ)

5 October 2022

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Instructions. *You must sign below to confirm that you have read, understand, and will follow them.*

- This is an 80-minute *closed-book* exam; no notes are allowed. Calculators are *not* permitted.
- The exam consists of 7 questions on 8 pages.
- Be sure to read carefully and follow the instructions for the individual problems. To receive full marks, your solution must be correct, complete, and show all relevant details.
- For rough work or additional work space, you may use the back pages. Do not use scrap paper of your own.
- Use proper mathematical notation and terminology.
- You may ask for clarification.
- Unauthorized electronic devices (such as cellular phones) are not permitted during this exam. Such devices must be *turned off completely* and stored out of students' reach. Students found in possession of such a device during the exam will be asked to leave immediately and academic fraud allegations may be filed.

LAST NAME: SOLUTIONS

First name: ______________________

Signature: ______________________

Write your student number on the next page.

Circle your DGD (this is where you will pick up your marked exam):

B01	B02	B03	B04
10:00	13:00	14:30	16:00
MRT 221	TBT 021	MRT 252	TBT 315

Student number:________________

Question	1	2	3	4	5	6	7	Total
Max	2	3	3	3	4	4	3	22
Marks								

[2pts] 1. Determine the domain of the following function. *Briefly explain your reasoning. Give your answer in the form of a union of intervals.*

$$g(x) = \frac{\sqrt{x^2 - 16}}{\ln(x+7)}$$

We need: $\begin{cases} x^2 - 16 \geq 0 & (1) \\ x + 7 > 0 & (2) \\ \ln(x+7) \neq 0 & (3) \end{cases}$

From (1): $x \in (-\infty, -4] \cup [4, \infty)$

(2): $x \in (-7, \infty)$

(3): $x \neq -6$

Hence the domain of g is

$(-7, -6) \cup (-6, -4] \cup [4, \infty)$

[3pts] 2. (a) Find all solutions to the equation $\log_2(x^2 - 2x) = 3$.

$$x^2 - 2x = 2^3$$
$$x^2 - 2x - 8 = 0$$
$$(x+2)(x-4) = 0$$
$$x_1 = -2$$
$$x_2 = 4$$

(b) Use (a) to find $f^{-1}(3)$ for the function $f(x) = \log_2(x) + \log_2(x-2)$.

Note: It can be verified that f is one-to-one, and hence invertible, but you need not show this.

Let $x = f^{-1}(3)$

then $f(x) = 3$

$$\log_2(x) + \log_2(x-2) = 3$$
$$\log_2(x(x-2)) = 3$$
$$\log_2(x^2 - 2x) = 3$$

By (a), we have $x \in \{-2, 4\}$

Since -2 is not in the domain of f,

we conclude $f^{-1}(3) = 4$.

3. Find the derivative of

$$f(x) = \sqrt{4x+1}$$

using the definition. *You may not use any of the differentiation rules from class, only the definition involving a limit. Show all relevant steps in your solution!*

[3pts]

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{4(x+h)+1} - \sqrt{4x+1}}{h}$$

$$= \lim_{h \to 0} \frac{(4(x+h)+1) - (4x+1)}{h\left(\sqrt{4(x+h)+1} + \sqrt{4x+1}\right)}$$

$$= \lim_{h \to 0} \frac{4h}{h\left(\sqrt{4(x+h)+1} + \sqrt{4x+1}\right)}$$

$$= \lim_{h \to 0} \frac{4}{\sqrt{4(x+h)+1} + \sqrt{4x+1}}$$

$$= \frac{4}{2\sqrt{4x+1}} = \frac{2}{\sqrt{4x+1}}$$

4. Let α and β be parameters, and define a function

$$g(x) = \begin{cases} \frac{\alpha}{x^2-1} & \text{if } x < 2 \\ \beta & \text{if } x = 2 \\ \frac{3x+\alpha}{4x-3} & \text{if } x > 2 \end{cases}.$$

[3pts] *Show all relevant steps when answering the following questions.*

(a) Determine $\lim_{x\to 2^-} g(x)$ and $\lim_{x\to 2^+} g(x)$.

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} \frac{\alpha}{x^2-1} = \frac{\alpha}{2^2-1} = \frac{\alpha}{3}$$

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \frac{3x+\alpha}{4x-3} = \frac{3\cdot 2+\alpha}{4\cdot 2-3} = \frac{6+\alpha}{5}$$

(b) Use your work in (a) to determine all values of α such that $\lim_{x\to 2} g(x)$ exists.

$$\lim_{x\to 2} g(x) \text{ exists } \Leftrightarrow \lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x)$$

$$\Leftrightarrow \quad \frac{\alpha}{3} = \frac{6+\alpha}{5} \quad \Leftrightarrow \quad \alpha = 9$$

(c) Use your work in (b) to determine all values of β such that g is continuous at 2.

$$g \text{ is continuous at } 2 \Leftrightarrow \lim_{x\to 2} g(x) = g(2)$$

$$\Leftrightarrow \quad \alpha = 9 \text{ and } \beta = \frac{\alpha}{3}$$

$$\Leftrightarrow \quad \alpha = 9 \text{ and } \beta = 3$$

5. Determine the following limits. *You may use any technique we have seen so far in the course. If you know L'Hospital's Rule — please do not use it. Show all relevant steps in your solution.*

[4pts]

(a) $\lim_{x\to\infty} \frac{2x^2+\sqrt{x^4-1}}{3x^2-4}$ $= \lim_{x\to\infty} \frac{2x^2 + x^2\sqrt{1-\frac{1}{x^4}}}{3x^2-4}$

$$= \lim_{x\to\infty} \frac{x^2\left(2+\sqrt{1-\frac{1}{x^4}}\right)}{x^2\left(3-\frac{4}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{2+\sqrt{1-\frac{1}{x^4}}}{3-\frac{4}{x^2}}$$

$$= \frac{2+1}{3-0} = \frac{3}{3} = 1$$

(b) $\lim_{x\to 4}\left(\frac{1}{x-4}-\frac{8}{x^2-16}\right)$ $= \lim_{x\to 4} \frac{(x+4)-8}{(x-4)(x+4)}$

$$= \lim_{x\to 4} \frac{x-4}{(x-4)(x+4)}$$

$$= \lim_{x\to 4} \frac{1}{x+4}$$

$$= \frac{1}{4+4} = \frac{1}{8}$$

6. Determine each of the following derivatives. *You may use any technique we have seen so*

[4pts] *far in the course. You do not need to simplify your answers.*

(a) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2t\cos t - 4\pi}{5t^4 + 3t}\right)$ $=$

$$= \frac{\frac{d}{dt}(2t\cos t - 4\pi)(5t^4+3t) - (2t\cos t - 4\pi)\frac{d}{dt}(5t^4+3t)}{(5t^4+3t)^2}$$

$$= \frac{(2\cos t - 2t\sin t)(5t^4+3t) - (2t\cos t - 4\pi)(20t^3+3)}{(5t^4+3t)^2}$$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan(4\sin(\sqrt{x}))\right)$ $=$

$$= \sec^2(4\sin\sqrt{x}) \cdot \frac{d}{dx}(4\sin\sqrt{x})$$

$$= \sec^2(4\sin\sqrt{x}) \cdot 4\cos\sqrt{x} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \sec^2(4\sin\sqrt{x}) \cdot 4\cos\sqrt{x} \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

$$= 2\sec^2(4\sin\sqrt{x})\cos\sqrt{x} \cdot \frac{1}{\sqrt{x}}$$

[3pts] 7. Determine the x-coordinates of all points on the curve $f(x) = (x+2)^2 e^{x-3}$ where the tangent line is horizontal. *Show all relevant steps in your solution.*

We need to find all x s.t. $f'(x) = 0$

$$f'(x) = 2(x+2)e^{x-3} + (x+2)^2 e^{x-3}$$

$$= (x+2)e^{x-3}(2 + x + 2)$$

$$= (x+2)(x+4)e^{x-3}$$

Hence $f'(x) = 0 \Rightarrow x = -2$ or $x = -4$