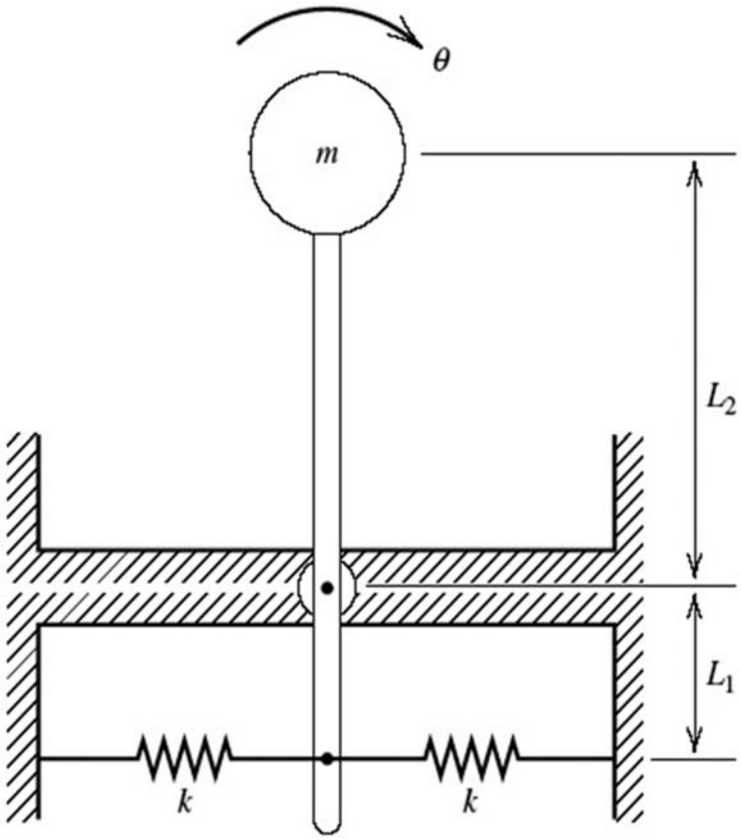
MCG4308 Midterm 2021

February 12, 2021

Question 1 - EofM

Consider the system shown in the figure that pivots around the fixed point. Use θ as the generalized coordinate for this system, with the positive direction being clockwise, as shown in the figure. You may make the following assumptions in your modelling: (i) the angle θ remains small (so you can use a small angle approximation). The bar connecting the springs and the mass *m* has negligible mass.



Solution

Consider the system shown in the figure that pivots around the fixed point. Use θ as the generalized coordinate for this system, with the positive direction being clockwise, as shown in the figure. You may make the following assumptions in your modelling: (i) the angle θ remains small (so you can use a small angle approximation) and (ii) the bar connecting the two springs and the mass *m* has negligible mass.

Newton

The moment of inertia of the bar+ mass system about the pivot point is given by

$J_o = m \cdot L_2^2$

$$J_o = m\, L_2^2 \tag{1.1.1}$$

The equation of motion is given by a sum of moments (due to the two springs and gravity) about the pivot point

$J_o \cdot \ddot{\theta} = M_{springs} + M_{gravity}$

$$J_o \left(\frac{d^2}{dt^2} \theta(t) \right) = M_{springs} + M_{gravity} \quad \textbf{(1.1.2)}$$

The springs are each stretched or compressed by $L_1 \cdot \theta$ (small angle assumption) and their moment arm is L_1. They both oppose the movement (ie tend to move θ in the negative direction for a displacment in the positive direction). Hence, the moment about the pivot point due to the two springs is given by

$M_{springs} = -2 \cdot k \cdot (L_1 \cdot \theta) \cdot L_1$

$$M_{springs} = -2\, k\, L_1^2\, \theta \quad \textbf{(1.1.3)}$$

The moment due to gravity is due to mass m - once displaced in the positive θ direction, it wants to keep displacement θ in the positive direction, so the moment is positive. The moment arm is $L_2 \cdot \sin(\theta) \approx L_2 \cdot \theta$ Hence:

$M_{gravity} = +(m \cdot g) \cdot L_2 \cdot \theta$

$$M_{gravity} = m\, g\, L_2\, \theta \quad \textbf{(1.1.4)}$$

Putting it all together into the equation of motion gives

subs((**1.1.1**), (**1.1.3**), (**1.1.4**), (**1.1.2**))

$$m\, L_2^2 \left(\frac{d^2}{dt^2} \theta(t) \right) = m\, g\, L_2\, \theta - 2\, k\, L_1^2\, \theta \quad \textbf{(1.1.5)}$$

Rearranging into standard form gives

collect(*lhs*((**1.1.5**)) − *rhs*((**1.1.5**)), *diff*) = 0

$$m\, L_2^2 \left(\frac{d^2}{dt^2} \theta(t) \right) - m\, g\, L_2\, \theta + 2\, k\, L_1^2\, \theta = 0 \quad \textbf{(1.1.6)}$$

Energies

Kinetic energy

The kinetic energy is only due to the mass *m* and, using the small angle assumption is given by

$T = \frac{1}{2} \cdot m \cdot (L_2 \cdot \dot{\theta})^2$

$$T = \frac{m\, L_2^2 \left(\frac{d}{dt} \theta(t) \right)^2}{2} \quad \textbf{(1.1.7)}$$

Hence the equivalent mass is given by

$m_{eq} = coeff\left(rhs(\textbf{(1.1.7)}), \dot{\theta}^2 \right) \cdot 2$

$$m_{eq} = m\, L_2^2 \quad \textbf{(1.1.8)}$$

Potential energy

The potential energy is due to the two springs + gravity

$V = V_{springs} + V_{gravity}$

$$V = V_{springs} + V_{gravity} \quad \textbf{(1.1.9)}$$

The potential energy due to the two springs is given by

$V_{springs} = \frac{1}{2} \cdot 2 \cdot k \cdot (L_1 \cdot \theta)^2$

$$V_{springs} = k\,L_1^2\,\theta^2 \quad \textbf{(1.1.10)}$$

The potential energy due to gravity is given by
$V_{gravity} = m \cdot g \cdot L_2 \cdot \cos(\theta)$

$$V_{gravity} = m\,g\,L_2\,\cos(\theta) \quad \textbf{(1.1.11)}$$

Using the small angle approximation
$subs\left(\cos(\theta) = 1 - \frac{\theta^2}{2}, \textbf{(1.1.11)}\right)$

$$V_{gravity} = m\,g\,L_2\left(1 - \frac{\theta^2}{2}\right) \quad \textbf{(1.1.12)}$$

From this we can put it all together
$subs(\textbf{(1.1.12)}, \textbf{(1.1.10)}, \textbf{(1.1.9)})$

$$V = k\,L_1^2\,\theta^2 + m\,g\,L_2\left(1 - \frac{\theta^2}{2}\right) \quad \textbf{(1.1.13)}$$

And now we can find the equivalent stiffness
$k_{eq} = coeff\left(rhs(\textbf{(1.1.13)}), \theta^2\right) \cdot 2$

$$k_{eq} = -m\,g\,L_2 + 2\,k\,L_1^2 \quad \textbf{(1.1.14)}$$

Question 2 - Find homogeneous and particular solution

$ode := 1 \cdot \frac{d^2}{dt^2}x(t) + (A+B) \cdot \frac{d}{dt}x(t) + (A \cdot B) \cdot x(t) = 2 \cdot A \cdot \left(A^2 + B^2\right) \cdot \cos(A \cdot t)$

$$ode := \frac{d^2}{dt^2}x(t) + (A+B)\left(\frac{d}{dt}x(t)\right) + A\,B\,x(t) = 2\,A\left(A^2 + B^2\right)\cos(A\,t) \quad \textbf{(2.1)}$$

Solution

Homogeneous solution - 'by hand'

The transient solution is the solution to (same ODE but zero on right hand side)

$ode2 := lhs(ode) = 0$

$$ode2 := \frac{d^2}{dt^2}x(t) + (A+B)\left(\frac{d}{dt}x(t)\right) + A\,B\,x(t) = 0 \quad \textbf{(2.2.1.1)}$$

For the transient/homogeneous solution, guess
$x_h(t) = \exp(s \cdot t)$

$$x_h(t) = e^{s\,t} \quad \textbf{(2.2.1.2)}$$

Substitute this into the equation of motion.

$subs(x(t) = \exp(s \cdot t), ode2)$

$$\frac{d^2}{dt^2} e^{st} + (A+B) \left(\frac{d}{dt} e^{st}\right) + A\,B\,e^{st} = 0 \qquad (2.2.1.3)$$

Simplify and divide through by the exponential to get the characteristic polynomial

$simplify\left(\frac{\mathbf{(2.2.1.3)}}{\exp(s \cdot t)}\right)$

$$(s+A)\,(s+B) = 0 \qquad (2.2.1.4)$$

Solve the characteristic polynomial

$S := solve(\mathbf{(2.2.1.4)}, s)$

$$S := -A, -B \qquad (2.2.1.5)$$

$x_h(t) = a_1 \cdot \exp(-A \cdot t) + a_2 \cdot \exp(-B \cdot t)$

$$x_h(t) = a_1\, e^{-A\,t} + a_2\, e^{-B\,t} \qquad (2.2.1.6)$$

For positive A and B, this solution will be overdamped

Particular solution - 'by hand'

Particular solution (response to the trig term)

Guess the form for the particular solution

$x_p(t) = (b_0 \cdot \cos(A \cdot t) + b_1 \cdot \sin(A \cdot t))$

$$x_p(t) = b_0 \cos(A\,t) + b_1 \sin(A\,t) \qquad (2.2.2.1)$$

Substitute this into the original differential equation with just the exponential term on the RHS

$subs(x(t) = rhs(\mathbf{(2.2.2.1)}), ode)$

$$\frac{\partial^2}{\partial t^2} \left(b_0 \cos(A\,t) + b_1 \sin(A\,t)\right) + (A+B) \left(\frac{\partial}{\partial t} \left(b_0 \cos(A\,t) + b_1 \sin(A\,t)\right)\right) + A\,B \left(b_0 \cos(A\,t) + b_1 \sin(A\,t)\right) = 2\,A\,(A^2 + B^2) \cos(A\,t) \qquad (2.2.2.2)$$

$collect(expand(\mathbf{(2.2.2.2)}), [\sin, \cos])$

$$\left(-A^2 b_0 - A^2 b_1 - A\,B\,b_0 + A\,B\,b_1\right) \sin(A\,t) + \left(-A^2 b_0 + A^2 b_1 + A\,B\,b_0 + A\,B\,b_1\right) \cos(A\,t) = \left(2\,A^3 + 2\,A\,B^2\right) \cos(A\,t) \qquad (2.2.2.3)$$

Matching up the coefficients of sines and cosines on the left and right hand sides:

$\left(-A^2 b_0 - A^2 b_1 - A\,B\,b_0 + A\,B\,b_1\right) = 0$

$$-A^2 b_0 - A^2 b_1 - A\,B\,b_0 + A\,B\,b_1 = 0 \qquad (2.2.2.4)$$

$\left(-A^2 b_0 + A^2 b_1 + A\,B\,b_0 + A\,B\,b_1\right) = \left(2\,A^3 + 2\,A\,B^2\right)$

$$-A^2 b_0 + A^2 b_1 + A\,B\,b_0 + A\,B\,b_1 = 2\,A^3 + 2\,A\,B^2 \qquad (2.2.2.5)$$

$solve(\{\mathbf{(2.2.2.4)}, \mathbf{(2.2.2.5)}\}, \{b_0, b_1\})$

$$\{b_0 = -A + B,\ b_1 = A + B\} \qquad (2.2.2.6)$$

This tells us we've found the particular solution (since it works)

$subs(\mathbf{(2.2.2.6)}, \mathbf{(2.2.2.1)})$

$$x_p(t) = (A+B) \sin(A\,t) + (-A+B) \cos(A\,t) \qquad (2.2.2.7)$$

homogeneous and particular solutions via Maple

dsolve(*ode*)

$$x(t) = e^{-Bt}\_C2 + e^{-At}\_C1 + (A+B)\sin(At) + (-A+B)\cos(At) \qquad \textbf{(2.2.3.1)}$$

Question 3 - find unknown coefficients

Question

The homogeneous solution to a particular differential equation is given by

$x_h(t) = e^{-At}\sin(2Bt)\,c_2 + e^{-At}\cos(2Bt)\,c_1$

$$x_h(t) = e^{-At}\sin(2Bt)\,c_2 + e^{-At}\cos(2Bt)\,c_1 \qquad \textbf{(3.1)}$$

And the particular solution is given by

$x_p(t) = 2\,e^{-2t}t + e^{-2t} + 2$

$$x_p(t) = 2\,e^{-2t}t + e^{-2t} + 2 \qquad \textbf{(3.2)}$$

The initial condions are given by

$x(0) = 3, \mathrm{D}(x)(0) = 4\cdot B$

$$x(0) = 3, \mathrm{D}(x)(0) = 4\,B \qquad \textbf{(3.3)}$$

Find c_1 and c_2 for the total solution of this problem.

Solution

$x(t) = subs\big(\textbf{(3.1)}, \textbf{(3.2)}, x_h(t) + x_p(t)\big)$

$$x(t) = e^{-At}\sin(2Bt)\,c_2 + e^{-At}\cos(2Bt)\,c_1 + 2\,e^{-2t}t + e^{-2t} + 2 \qquad \textbf{(3.4)}$$

$simplify(subs(t=0, rhs(\textbf{(3.4)}))) = 3$

$$3 + c_1 = 3 \qquad \textbf{(3.5)}$$

$simplify(subs(t=0, diff(rhs(\textbf{(3.4)}), t))) = 4\cdot B$

$$-A\,c_1 + 2\,B\,c_2 = 4\,B \qquad \textbf{(3.6)}$$

$solve\big(\{\textbf{(3.5)}, \textbf{(3.6)}\}, \{c_1, c_2\}\big)$

$$\{c_1 = 0, c_2 = 2\} \qquad \textbf{(3.7)}$$

Part II - multiple choice

Question i

Find the maximum acceleration of the very lightly damped system that is described by

$x(t) = \exp\left(-\frac{t}{100}\right)\cdot\sin(B\cdot t) + \exp\left(-\frac{t}{100}\right)\cdot\cos(B\cdot t)$

$$x(t) = e^{-\frac{t}{100}}\sin(Bt) + e^{-\frac{t}{100}}\cos(Bt) \qquad \textbf{(4.1)}$$

Solution

Since the system is very lightly damped (you are told this plus the exponential term has a very slow decay rate (time constant)). So we can use the no damping formula to find the maximum in acceleration

$x(t) = A_1 \cdot \sin(B \cdot t) + A_2 \cdot \cos(B \cdot t)$ can be written as $x(t) = A \cdot \sin(B \cdot t + \phi)$ where $A = \sqrt{A_1^2 + A_2^2}$

Then for an undamped/lightly damped system, $a_{max} = A \cdot B^2$

Here we have

$$x(t) = \exp\left(-\frac{t}{100}\right) \cdot \sin(B \cdot t) + \exp\left(-\frac{t}{100}\right) \cdot \cos(B \cdot t)$$

$$x(t) = e^{-\frac{t}{100}} \sin(t B) + e^{-\frac{t}{100}} \cos(t B) \tag{4.1.1}$$

Here $A_1 = 1, A_2 = 1$ so $A = \sqrt{2}$. So $a_{max} = \sqrt{2} \cdot B^2 = a_{max} = \sqrt{2}\, B^2$

Question ii

Kinetic energy of the wheel rolling without slip is given by
(a) correct

$$T = \frac{1}{2} \cdot \left(m r^2 + J_G\right) \left(\frac{d}{dt} \theta(t)\right)^2$$

$$T = \frac{\left(m r^2 + J_G\right) \left(\frac{d}{dt} \theta(t)\right)^2}{2} \tag{5.1}$$

(b) Incorrect

$$T = \frac{1}{2} \cdot \left(m + \frac{J_G}{r^2}\right) \left(\frac{d}{dt} \theta(t)\right)^2$$

$$T = \frac{\left(m + \frac{J_G}{r^2}\right) \left(\frac{d}{dt} \theta(t)\right)^2}{2} \tag{5.2}$$

(c) Incorrect

$$T = \frac{1}{2} \cdot (m) \left(\frac{d}{dt} x(t)\right)^2$$

$$T = \frac{m \left(\frac{d}{dt} x(t)\right)^2}{2} \tag{5.3}$$

(d) Incorrect

$$T = \frac{(J_G) \left(\frac{d}{dt} \theta(t)\right)^2}{2}$$

$$T = \frac{J_G \left(\frac{d}{dt} \theta(t)\right)^2}{2} \tag{5.4}$$

For a wheel of radius r, mass m and moment of inertia J_G about its centre of mass rolls without slip. The kinetic energy is given by

$$T=\frac{1}{2}\cdot m\cdot\dot{x}^2+\frac{1}{2}\cdot J_G\cdot\dot{\theta}^2$$

$$T=\frac{m\left(\frac{d}{dt}x(t)\right)^2}{2}+\frac{J_G\left(\frac{d}{dt}\theta(t)\right)^2}{2} \qquad \textbf{(5.1.1)}$$

Using the no slip condition of $x=r\cdot\theta$ then the kinetic energy in terms of the linear displacement variable

$$collect\left(subs\left(\dot{\theta}=\frac{\dot{x}}{r},\ \textbf{(5.1.1)}\right),\ \dot{x}\right)$$

$$T=\left(\frac{m}{2}+\frac{J_G}{2\,r^2}\right)\left(\frac{d}{dt}x(t)\right)^2 \qquad \textbf{(5.1.2)}$$

Using the no slip condition of $x=r\cdot\theta$ then the kinetic energy in terms of the angular displacement variable

$$collect\left(subs\left(\dot{x}=r\cdot\dot{\theta},\ \textbf{(5.1.1)}\right),\ \dot{\theta}\right)$$

$$T=\left(\frac{m\,r^2}{2}+\frac{J_G}{2}\right)\left(\frac{d}{dt}\theta(t)\right)^2 \qquad \textbf{(5.1.3)}$$

Question iii

What do you need to add to $Z1 := (a+b\cdot I)\cdot\exp((-A+B\cdot I)\cdot t)$: so that the result is real (ie has no imaginary part)

(a) Answer: add the complex conjugate

$ans1 := (a-b\cdot I)\cdot\exp((-A-B\cdot I)\cdot t)$

$$ans1 := (a-\mathrm{I}\,b)\,e^{(-A-\mathrm{I}B)\,t} \qquad \textbf{(6.1)}$$

$ans1+Z1$

$$(a+\mathrm{I}\,b)\,e^{(-A+\mathrm{I}B)\,t}+(a-\mathrm{I}\,b)\,e^{(-A-\mathrm{I}B)\,t} \qquad \textbf{(6.2)}$$

$evalc(\textbf{(6.2)})$

$$2\,a\,e^{-tA}\cos(t\,B)-2\,b\,e^{-tA}\sin(t\,B) \qquad \textbf{(6.3)}$$

(b) Incorrect

$ans2 := (a-b\cdot I)\cdot\exp((+A-B\cdot I)\cdot t)$

$$ans2 := (a-\mathrm{I}\,b)\,e^{(-\mathrm{I}B+A)\,t} \qquad \textbf{(6.4)}$$

$ans2+Z1$

$$(a-\mathrm{I}\,b)\,e^{(-\mathrm{I}B+A)\,t}+(a+\mathrm{I}\,b)\,e^{(-A+\mathrm{I}B)\,t} \qquad \textbf{(6.5)}$$

$evalc(\textbf{(6.5)})$

$$a\,e^{tA}\cos(t\,B)-b\,e^{tA}\sin(t\,B)+a\,e^{-tA}\cos(t\,B)-b\,e^{-tA}\sin(t\,B)+\mathrm{I}\left(-b\,e^{tA}\cos(t\,B)-a\,e^{tA}\sin(t\,B)+b\,e^{-tA}\cos(t\,B)+a\,e^{-tA}\sin(t\,B)\right) \qquad \textbf{(6.6)}$$

(c) Incorrect

$ans3 := (a + b \cdot I) \cdot \exp((-A - B \cdot I) \cdot t)$

$$ans3 := (a + \mathrm{I}\,b)\, \mathrm{e}^{(-A - \mathrm{I}B)\,t} \tag{6.7}$$

$ans3 + Z1$

$$(a + \mathrm{I}\,b)\, \mathrm{e}^{(-A - \mathrm{I}B)\,t} + (a + \mathrm{I}\,b)\, \mathrm{e}^{(-A + \mathrm{I}B)\,t} \tag{6.8}$$

$evalc(\mathbf{(6.8)})$

$$2\,a\,\mathrm{e}^{-tA} \cos(t\,B) + 2\,\mathrm{I}\,b\,\mathrm{e}^{-tA} \cos(t\,B) \tag{6.9}$$

(d) Incorrect

$ans4 := (a - b \cdot I) \cdot \exp((-A + B \cdot I) \cdot t)$

$$ans4 := (a - \mathrm{I}\,b)\, \mathrm{e}^{(-A + \mathrm{I}B)\,t} \tag{6.10}$$

$ans4 + Z1$

$$(a - \mathrm{I}\,b)\, \mathrm{e}^{(-A + \mathrm{I}B)\,t} + (a + \mathrm{I}\,b)\, \mathrm{e}^{(-A + \mathrm{I}B)\,t} \tag{6.11}$$

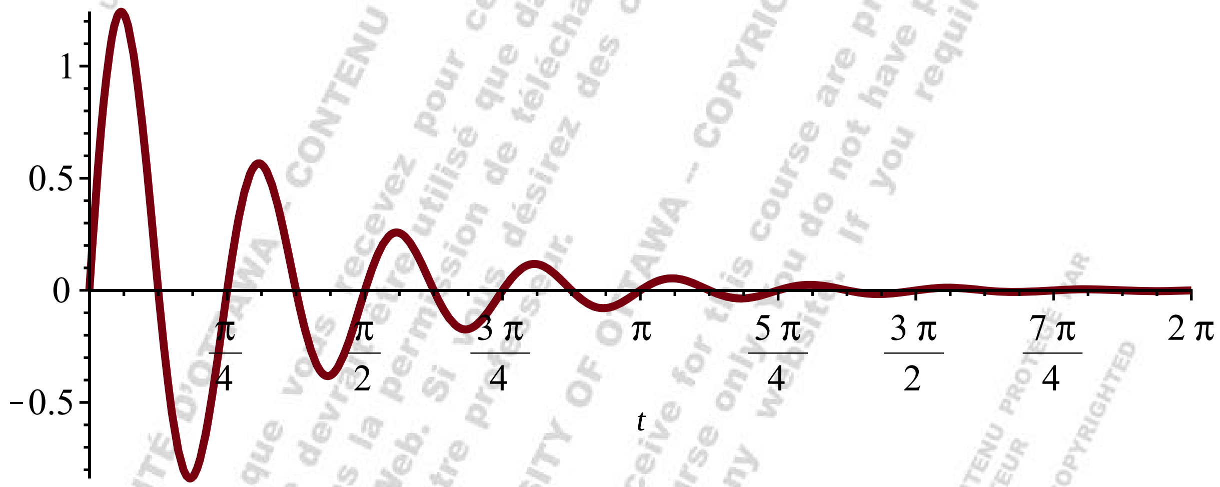
$evalc(\mathbf{(6.11)})$

$$2\,a\,\mathrm{e}^{-tA} \cos(t\,B) + 2\,\mathrm{I}\,a\,\mathrm{e}^{-tA} \sin(t\,B) \tag{6.12}$$

Question iv

The homogeneous response to a particular ordinary differential equation looks like

$plot(1.5 \cdot \exp(-t) \cdot \sin(8 \cdot t), t = 0..2 \cdot \mathrm{Pi}, thickness = 3)$



The roots of the characteristic equation of this differential equation must be

(a) ANS: complex conjugate pair with negative real part e.g. $-2 \pm bi$

(b) complex conjugate pair with positive real part e.g. $+2 \pm bi$

(c) two real positive roots e.g. 2,3

(d) two real negative roots e.g. -2,-3

(e) none of the other responses are correct

Question v

Some given system has mass m, damping c and stiffness k. The generalized coordinate for the system is $x(t)$, and the system is subject to an external forcing input $y(t)$. The equation of motion is given by $m \cdot \ddot{x} + c \cdot \dot{x} + k \cdot x = k \cdot y + k \cdot \dddot{y}$

$$m \left(\frac{d^2}{dt^2} x(t) \right) + c \left(\frac{d}{dt} x(t) \right) + k\,x(t) = k\,y(t) + k \left(\frac{d^3}{dt^3} y(t) \right) \tag{8.1}$$

$with(inttrans):$

$laplace(\mathbf{(8.1)}, t, s)$

$$\mathcal{L}(x(t), t, s)\, s^2\, m + \mathcal{L}(x(t), t, s)\, s\, c - x(0)\, s\, m + k\,\mathcal{L}(x(t), t, s) - \mathrm{D}(x)(0)\, m - x(0)\, c = \tag{8.2}$$

$$-k\left(D^{(2)}(y)(0)+s\,D(y)(0)+s^2\,y(0)\right)+\mathcal{L}(y(t),t,s)\,s^3\,k+k\,\mathcal{L}(y(t),t,s)$$

collect(*subs*($x(0)=0$, $D(x)(0)=0$, $y(0)=0$, $D(y)(0)=0$, $D^{(2)}(y)(0)=0$, **(8.2)**), *laplace*)

$$\left(m\,s^2+c\,s+k\right)\mathcal{L}(x(t),t,s)=\left(k\,s^3+k\right)\mathcal{L}(y(t),t,s) \qquad \textbf{(8.3)}$$

laplace($x(t)$, t, s) = *solve*(**(8.3)**, *laplace*($x(t)$, t, s))

$$\mathcal{L}(x(t),t,s)=\frac{k\left(s^3+1\right)\mathcal{L}(y(t),t,s)}{m\,s^2+c\,s+k} \qquad \textbf{(8.4)}$$

$$H(s)=\frac{laplace(x(t),t,s)}{laplace(y(t),t,s)}$$

$$H(s)=\frac{\mathcal{L}(x(t),t,s)}{\mathcal{L}(y(t),t,s)} \qquad \textbf{(8.5)}$$

subs(**(8.4)**, **(8.5)**)

$$H(s)=\frac{k\left(s^3+1\right)}{m\,s^2+c\,s+k} \qquad \textbf{(8.6)}$$

(a) correct answer - NB - you can do this by inspection as long as you remember the rules of laplace transform of derivatives of functions
(8.6)

$$H(s)=\frac{k\left(s^3+1\right)}{m\,s^2+c\,s+k} \qquad \textbf{(8.7)}$$

(b) Incorrect

$$H(s)=\frac{k\left(s^3+s\right)}{m\,s^2+c\,s+k}$$

$$H(s)=\frac{k\left(s^3+s\right)}{m\,s^2+c\,s+k} \qquad \textbf{(8.8)}$$

(c) Incorrect

$$H(s)=\frac{m\,s^2+c\,s+k}{k\left(s^3+1\right)}$$

$$H(s)=\frac{m\,s^2+c\,s+k}{k\left(s^3+1\right)} \qquad \textbf{(8.9)}$$

(d) Incorrect

$$H(s)=\frac{m\,s^2+c\,s+k}{k\left(s^3+s\right)}$$

$$H(s)=\frac{m\,s^2+c\,s+k}{k\left(s^3+s\right)} \qquad \textbf{(8.10)}$$

Question vi

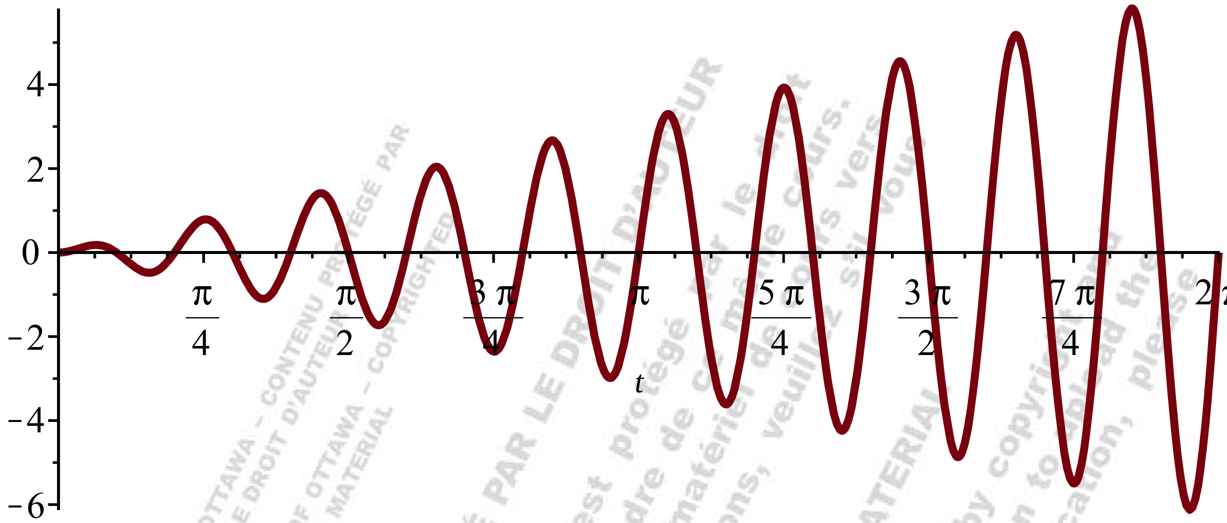
Consider a given mass-spring single degree of freedom system with equation of motion given by $m\cdot\ddot{x}+k\cdot x=f(t)$

$$m\left(\frac{d^2}{dt^2}x(t)\right)+k\,x(t)=f(t) \qquad \textbf{(9.1)}$$

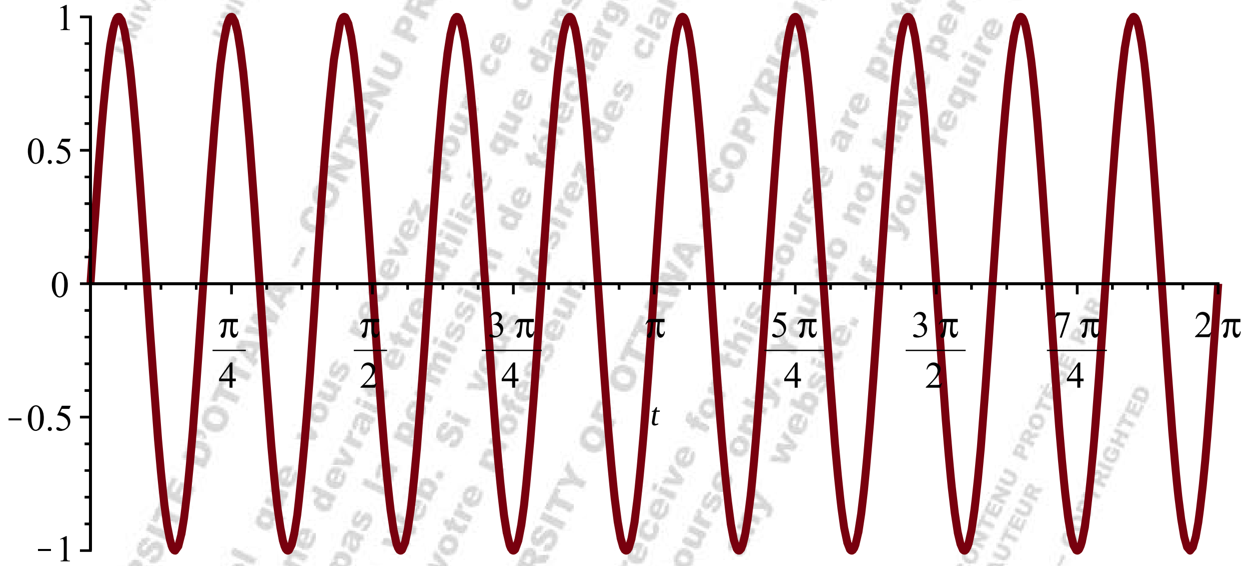
Suppose that the is subject to a sinusoidal forcing function $f(t)=f_0\cdot\cos\left(\sqrt{\frac{k}{m}}\cdot t\right)$

Which of the following responses best describe the system response to this forcing function?

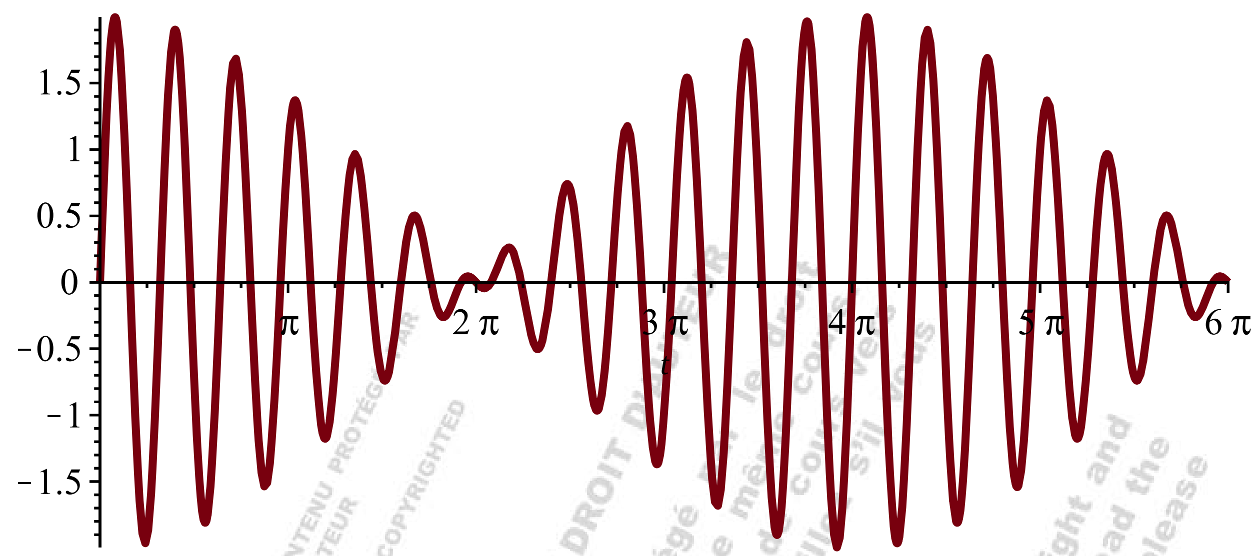
(a) correct answer
$plot(t \cdot \sin(10\, t),\ t = 0\,..2 \cdot \mathrm{Pi},\ thickness = 3)$



(b) Incorrect
$plot(\sin(10\, t),\ t = 0\,..2 \cdot \mathrm{Pi},\ thickness = 3)$

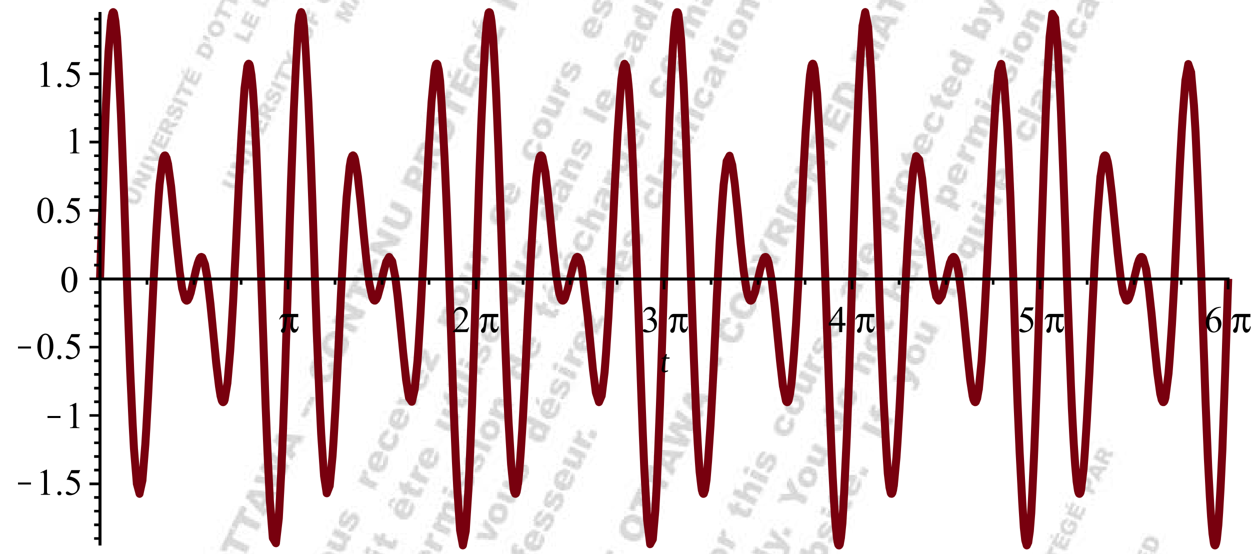


(c) Incorrect
$plot(\sin(6\, t) + \sin(6.5 \cdot t),\ t = 0\,..6 \cdot \mathrm{Pi},\ thickness = 3)$



(d) Incorrect

plot(sin(6 *t*) + sin(8·*t*), *t* = 0 ..6·Pi, *thickness* = 3)



Question vii

Suppose that a given mass-spring-damper system is critically damped. Which of the following is true about the damping ratio?

(a) $\zeta = 1$ (answer)

(b) $\zeta < 1$

(c) $\zeta > 1$

(d) $\zeta = 0$

Question viii

The complex frequency response of a given system is given by

$$T(r) = \frac{(1 + r \cdot I)}{1 - r^2 + 2 \cdot \zeta \cdot r \cdot I}$$

$$T(r) = \frac{1 + \mathrm{I}\, r}{1 - r^2 + 2\,\mathrm{I}\,\zeta\, r} \qquad \textbf{(11.1)}$$

Find the magnitude of the frequency response.

assume(*r*, *real*)
assume(ξ, *real*)
(a) correct response

$$|T(r)| = \text{abs}\left(\frac{(1+r\cdot I)}{1-r^2+2\cdot\xi\cdot r\cdot I}\right)$$

$$|T(r)| = \frac{\sqrt{r^2+1}}{\sqrt{(r^2-1)^2+4\,\xi^2\,r^2}} \quad \textbf{(11.1.1)}$$

Question ix

Suppose that a given mass-spring-damper system has a damping ratio given by

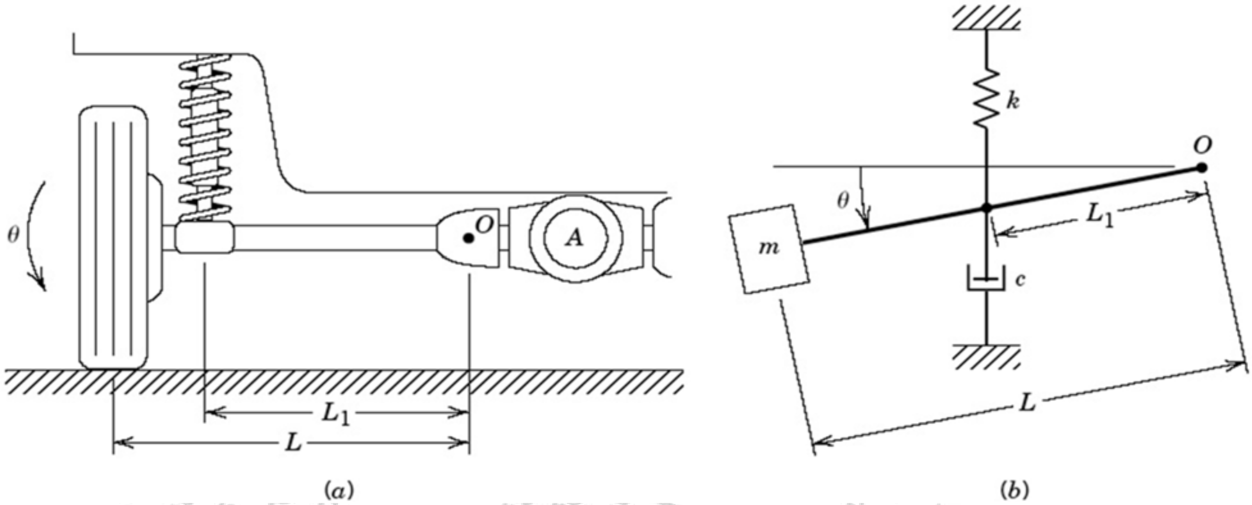
$$\zeta_{orig} = \frac{c}{2\cdot\sqrt{m\cdot k}} = 1$$

Quadruple the mass so that the 'new' system has mass $4\cdot m$
The new system will have damping ratio given by

$$\zeta_{new} = \frac{c}{2\cdot\sqrt{4\cdot m\cdot k}} = \frac{1}{2}\cdot\zeta_{orig} = \frac{1}{2}$$

The new system is now underdamped.

Question x



Equation of motion
Assuming small displacements, the equation of motion for the system shown in the figure is given by

$$(m\cdot L^2)\cdot\ddot{\theta} = -k\cdot(L_1\cdot\theta)\cdot L_1 - c\cdot(L_1\cdot\dot{\theta})\cdot L_1 + m\cdot g\cdot L$$

Find the natural frequency of this system

Solution

Equation of motion
Assuming small displacements, the equation of motion for the system shown in the figure is given by

$(m \cdot L^2) \cdot \ddot{\theta} = -k \cdot (L_1 \cdot \theta) \cdot L_1 - c \cdot (L_1 \cdot \dot{\theta}) \cdot L_1 + m \cdot g \cdot L$

$$m L^2 \left(\frac{d^2}{dt^2} \theta(t) \right) = -k L_1^2 \theta - c L_1^2 \left(\frac{d}{dt} \theta(t) \right) + m g L \qquad \textbf{(13.1.1)}$$

$m L^2 \left(\frac{d^2}{dt^2} \theta(t) \right) + c L_1^2 \left(\frac{d}{dt} \theta(t) \right) + k L_1^2 \theta(t) = m g L$

$$m L^2 \left(\frac{d^2}{dt^2} \theta(t) \right) + c L_1^2 \left(\frac{d}{dt} \theta(t) \right) + k L_1^2 \theta(t) = m g L \qquad \textbf{(13.1.2)}$$

The equivalent stiffness (coefficient of the θ term) is

$k_{eq} = k L_1^2$

$$k_{eq} = k L_1^2 \qquad \textbf{(13.1.3)}$$

The equivalent mass (coefficient of the $\frac{d^2}{dt^2} \theta(t)$ term) is

$m_{eq} = m L^2$

$$m_{eq} = m L^2 \qquad \textbf{(13.1.4)}$$

Hence, the natural frequency of this system is given by

$\omega_n = \mathrm{sqrt}\left(\frac{k_{eq}}{m_{eq}} \right)$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} \qquad \textbf{(13.1.5)}$$

subs(**(13.1.3)**, **(13.1.4)**, **(13.1.5)**)

$$\omega_n = \sqrt{\frac{k L_1^2}{m L^2}} \qquad \textbf{(13.1.6)}$$

Question xi

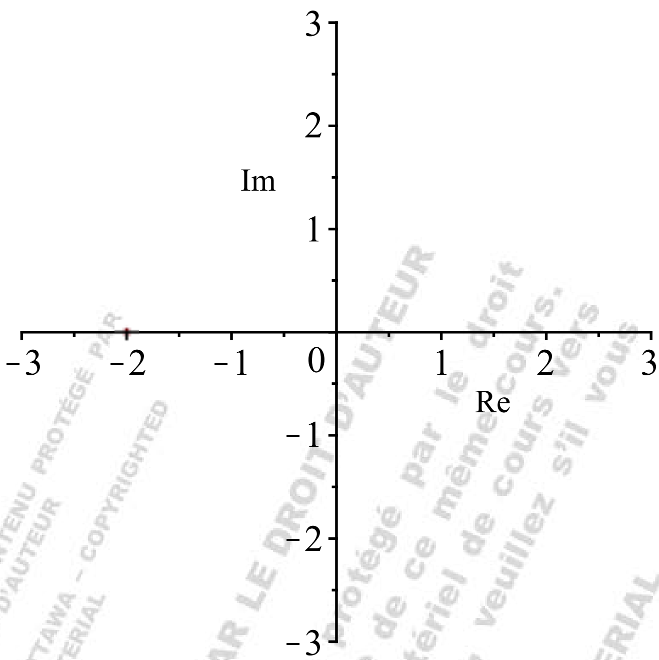
Any real,positive number $z = -R$ can be written in the form $z = R \cdot \exp(\pi)$ (meaning, modulus/radius R and phase of π)

assume(*R*, *positive*)
convert(−*R*, polar)

$$\mathrm{polar}(R, \pi) \qquad \textbf{(14.1)}$$

This may not be obvious on first pass, if so, plotting the numer in the complex plane always helps.

with(*plots*) :
complexplot([−2], *x* = −3 ..3, *y* = −3 ..3, *labels* = ["Re", "Im"], *style* = *point*,)



Question xii - frequency response

Some given system has mass m, damping c and stiffness k. The generalized coordinate for the system is $x(t)$, and the system is subject to an external forcing input $y(t)$. The equation of motion is given by

$m\cdot\ddot{x} + c\cdot\dot{x} + k\cdot x = k\cdot y(t)$

$$m\left(\frac{d^2}{dt^2}x(t)\right) + c\left(\frac{d}{dt}x(t)\right) + k\,x(t) = k\,y(t) \tag{15.1}$$

Find the complex frequency response to a sinusoidal input of the form $y(t) = \exp(I\cdot\omega\cdot t)$.

$H(s) := \dfrac{k}{m\cdot s^2 + c\cdot s + k}$

$$H := s \mapsto \frac{k}{m\cdot s^2 + c\cdot s + k} \tag{15.1.1}$$

$T(\omega) = H(I\cdot\omega)$

$$T(\omega) = \frac{k}{-m\,\omega^2 + c\,\omega\,\mathrm{I} + k} \tag{15.1.2}$$