ELG3175
Introduction to Communication Systems

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Solutions

Assignment #2

#1 (a) The spectrum (FT) of $\Pi(t)$ is

$$S_\Pi(f) = FT\{\Pi(t)\} = \text{sinc}(f)$$

Using linearity & scaling properties:

$$FT\{\varepsilon^{-1}\Pi(\frac{t}{\varepsilon})\} = \varepsilon^{-1} \cdot \varepsilon \cdot \text{sinc}(\varepsilon f) = \text{sinc}(\varepsilon f)$$

Hence,

$$FT\{\delta(t)\} = \lim_{\varepsilon \to 0} \text{sinc}(\varepsilon f) = \underline{1}.$$

(b) The spectrum of $\text{sinc}(t)$ is

$$FT\{\text{sinc}(t)\} = \Pi(f)$$

Using linearity & scaling properties:

$$FT\{\varepsilon^{-1}\text{sinc}(t\varepsilon^{-1})\} = \varepsilon^{-1} \cdot \varepsilon \cdot \Pi(f \cdot \varepsilon) = \Pi(f\varepsilon)$$

Finally,

$$FT\{\delta(t)\} = \lim_{\varepsilon \to 0} \Pi(\varepsilon f) = \underline{1}.$$

Problem 2[redacted]

a) We can write $x(t)$ as $x(t) = 2\Pi(\frac{t}{4}) - 2\Lambda(\frac{t}{2})$. Then

$$\mathcal{F}[x(t)] = \mathcal{F}[2\Pi(\frac{t}{4})] - \mathcal{F}[2\Lambda(\frac{t}{2})] = 8\text{sinc}(4f) - 4\text{sinc}^2(2f)$$

d) We can write $x(t)$ as $x(t) = \Lambda(t+1) - \Lambda(t-1)$. Thus

$$X(f) = \text{sinc}^2(f)e^{j2\pi f} - \text{sinc}^2(f)e^{-j2\pi f} = 2j\text{sinc}^2(f)\sin(2\pi f)$$

e) We can write $x(t)$ as $x(t) = \Lambda(t+1) + \Lambda(t) + \Lambda(t-1)$. Hence,

$$X(f) = \text{sinc}^2(f)(1 + e^{j2\pi f} + e^{-j2\pi f}) = \text{sinc}^2(f)(1 + 2\cos(2\pi f)$$

[redacted]

Problem [redacted] 3

We have

$$\begin{aligned}
\mathcal{F}[x(t) \star y(t)] &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,d\tau\right] e^{-j2\pi f t} dt \\
&= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(t-\tau)e^{-j2\pi f(t-\tau)}\,dt\right] e^{-j2\pi f\tau} d\tau
\end{aligned}$$

Now with the change of variable $u = t - \tau$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} y(t-\tau)e^{-j2\pi f(t-\tau)} dt &= \int_{-\infty}^{\infty} y(u)e^{-j2\pi f u} du \\
&= \mathcal{F}[y(t)] \\
&= Y(f)
\end{aligned}$$

and, therefore,

$$\begin{aligned}
\mathcal{F}[x(t) \star y(t)] &= \int_{-\infty}^{\infty} x(\tau)Y(f)e^{-j2\pi f\tau} d\tau \\
&= X(f) \cdot Y(f)
\end{aligned}$$

Problem [redacted] 4
(Convolution theorem:)

$$\mathcal{F}[x(t) \star y(t)] = \mathcal{F}[x(t)]\mathcal{F}[y(t)] = X(f)Y(f)$$

Thus

$$\begin{aligned} \text{sinc}(t) \star \text{sinc}(t) &= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t) \star \text{sinc}(t)]] \\ &= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t)] \cdot \mathcal{F}[\text{sinc}(t)]] \\ &= \mathcal{F}^{-1}[\Pi(f)\Pi(f)] = \mathcal{F}^{-1}[\Pi(f)] \\ &= \text{sinc}(t) \end{aligned}$$

Problem [redacted] 5
1) Clearly

$$\begin{aligned} x_1(t + kT_0) &= \sum_{n=-\infty}^{\infty} x(t + kT_0 - nT_0) = \sum_{n=-\infty}^{\infty} x(t - (n-k)T_0) \\ &= \sum_{m=-\infty}^{\infty} x(t - mT_0) = x_1(t) \end{aligned}$$

where we used the change of variable $m = n - k$.

2)

$$x_1(t) = x(t) \star \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

This is because

$$\int_{-\infty}^{\infty} x(\tau) \sum_{n=-\infty}^{\infty} \delta(t - \tau - nT_0) d\tau = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau - nT_0) d\tau = \sum_{n=-\infty}^{\infty} x(t - nT_0)$$

3)

$$\begin{aligned} \mathcal{F}[x_1(t)] &= \mathcal{F}[x(t) \star \sum_{n=-\infty}^{\infty} \delta(t - nT_0)] = \mathcal{F}[x(t)]\mathcal{F}[\sum_{n=-\infty}^{\infty} \delta(t - nT_0)] \\ &= X(f)\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_0}) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} X(\frac{n}{T_0})\delta(f - \frac{n}{T_0}) \end{aligned}$$

Problem [redacted] 6

1)

$$
\begin{aligned}
E_{x_1} &= \int_{-\infty}^{\infty} x_1^2(t)\,dt \\
&= \int_0^{\infty} e^{-2t}\cos^2 t\,dt \\
&< \int_0^{\infty} e^{-2t}\,dt = \frac{1}{2}
\end{aligned}
$$

where we have used $\cos^2 t \le 1$. Therefore, $x_1(t)$ is energy type. To find the energy we have

$$
\begin{aligned}
\int_0^{\infty} e^{-2t}\cos^2 t\,dt &= \frac{1}{2}\int_0^{\infty} e^{-2t}\,dt + \frac{1}{2}\int_0^{\infty} e^{-2t}\cos 2t\,dt \\
&= \frac{1}{4} + \frac{1}{2}\left[-\frac{1}{4}e^{-2t}\cos(2t) + \frac{1}{4}e^{-2*t}\sin(2t)\right]_0^{\infty} \\
&= \frac{3}{8}
\end{aligned}
$$

2)

$$
\begin{aligned}
E_{x_2} = \int_{-\infty}^{\infty} x_2^2(t)\,dt &= \int_{-\infty}^{0} e^{-2t}\cos^2(t)\,dt + \int_0^{\infty} e^{-2t}\cos^2(t)\,dt \\
&= \int_0^{\infty} e^{2t}\cos^2(t)\,dt + \frac{3}{8} \\
&= \frac{3}{8} - \frac{3}{8} + \lim_{t\to\infty}\frac{e^{2t}}{8}\left(2\cos^2(t) + \sin(2t) + 1\right) \\
&= \lim_{t\to\infty}\frac{e^{2t}}{8}f(t)
\end{aligned}
$$

where $f(t) = 2\cos^2(t) + \sin(2t) + 1$. By taking the derivative and setting it equal to zero we can find the minimum of $f(t)$ and show that $f(t) > 0.5$. This shows that $\lim_{t\to\infty}\frac{e^{2t}}{8}f(t) \ge \lim_{t\to\infty}\frac{e^{2t}}{16} = \infty$. This shows that the signal is not energy-type.

To check if the signal is power type, we obviously have $\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-2t}\cos^2 t\,dt = 0$. Therefore

$$
\begin{aligned}
P &= \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{2t}\cos^2(t)\,dt \\
&= \lim_{T\to\infty}\frac{1/4\,e^{2T}(\cos(T))^2 + 1/4\,e^{2T}\cos(T)\sin(T) + 1/8\left(e^T\right)^2 - 3/8}{T} \\
&= \infty
\end{aligned}
$$

Therefore $x_2(t)$ is neither power- nor energy-type.

3)

$$
\begin{aligned}
E_{x_3} = \int_{-\infty}^{\infty}(\mathrm{sgn}(t))^2\,dt &= \int_{-\infty}^{\infty} 1\,dt \\
&= \infty
\end{aligned}
$$

and hence the signal is not energy-type. To find the power

$$
\begin{aligned}
P_{x_3} &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(\mathrm{sgn}(t))^2\,dt \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}1^2\,dt \\
&= \lim_{T\to\infty}\frac{1}{2T}2T = 1
\end{aligned}
$$

4) Since $x_4(t)$ is periodic (or almost periodic when f_1/f_2 is not rational) the signal is not energy type. To see whether it is power type, we have

$$\begin{aligned}
P_{x_4} &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (A\cos 2\pi f_1 t + B\cos 2\pi f_2 t)^2\, dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left(A^2\cos^2 2\pi f_1 t + B^2\cos^2 2\pi f_2 t + 2AB\cos 2\pi f_1 t\cos 2\pi f_2 t\right) dt \\
&= \frac{A^2+B^2}{2}
\end{aligned}$$

Problem [redacted] 8

1) $x(t) = e^{-\alpha t}u_{-1}(t)$. The spectrum of the signal is $X(f) = \frac{1}{\alpha + j2\pi f}$ and the energy spectral density

$$\mathcal{G}_X(f) = |X(f)|^2 = \frac{1}{\alpha^2 + 4\pi^2 f^2}$$

Thus,

$$R_X(\tau) = \mathcal{F}^{-1}[\mathcal{G}_X(f)] = \frac{1}{2\alpha}e^{-\alpha|\tau|}$$

The energy content of the signal is

$$E_X = R_X(0) = \frac{1}{2\alpha}$$

2) $x(t) = \text{sinc}(t)$. Clearly $X(f) = \Pi(f)$ so that $\mathcal{G}_X(f) = |X(f)|^2 = \Pi^2(f) = \Pi(f)$. The energy content of the signal is

$$E_X = \int_{-\infty}^{\infty} \Pi(f)df = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Pi(f)df = 1$$

3) $x(t) = \sum_{n=-\infty}^{\infty} \Lambda(t-2n)$. The signal is periodic and thus it is not of the energy type. The power content of the signal is

$$\begin{aligned} P_x &= \frac{1}{2}\int_{-1}^{1} |x(t)|^2 dt = \frac{1}{2}\int_{-1}^{0} (t+1)^2 dt + \int_{0}^{1} (-t+1)^2 dt \\ &= \frac{1}{2}\left(\frac{1}{3}t^3 + t^2 + t\right)\Big|_{-1}^{0} + \frac{1}{2}\left(\frac{1}{3}t^3 - t^2 + t\right)\Big|_{0}^{1} \\ &= \frac{1}{3} \end{aligned}$$

The same result is obtain if we let

$$\mathcal{S}_X(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta(f - \frac{n}{2})$$

with $x_0 = \frac{1}{2}$, $x_{2l} = 0$ and $x_{2l+1} = \frac{2}{\pi(2l+1)}$ (see Problem 2.2). Then

$$\begin{aligned} P_X &= \sum_{n=-\infty}^{\infty} |x_n|^2 \\ &= \frac{1}{4} + \frac{8}{\pi^2}\sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} = \frac{1}{4} + \frac{8}{\pi^2}\frac{\pi^2}{96} = \frac{1}{3} \end{aligned}$$

4)

$$E_X = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u_{-1}(t)|^2 dt = \lim_{T\to\infty} \int_{0}^{\frac{T}{2}} dt = \lim_{T\to\infty} \frac{T}{2} = \infty$$

Thus, the signal is not of the energy type.

$$P_X = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u_{-1}(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\frac{T}{2} = \frac{1}{2}$$

Hence, the signal is of the power type and its power content is $\frac{1}{2}$. To find the power spectral density we find first the autocorrelation $R_X(\tau)$.

$$\begin{aligned} R_X(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u_{-1}(t)u_{-1}(t-\tau)dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{\tau}^{\frac{T}{2}} dt \\ &= \lim_{T\to\infty} \frac{1}{T}(\frac{T}{2} - \tau) = \frac{1}{2} \end{aligned}$$

Thus, $\mathcal{S}_X(f) = \mathcal{F}[R_X(\tau)] = \frac{1}{2}\delta(f)$.

5) Clearly $|X(f)|^2 = \pi^2 \mathrm{sgn}^2(f) = \pi^2$ and $E_X = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \pi^2 dt = \infty$. The signal is not of the energy type for the energy content is not bounded. Consider now the signal

$$x_T(t) = \frac{1}{t}\Pi(\frac{t}{T})$$

Then,

$$X_T(f) = -j\pi \mathrm{sgn}(f) \star T\mathrm{sinc}(fT)$$

and

$$S_X(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} = \lim_{T\to\infty} \pi^2 T \left| \int_{-\infty}^{f} \mathrm{sinc}(vT)dv - \int_{f}^{\infty} \mathrm{sinc}(vT)dv \right|^2$$

However, the squared term on the right side is bounded away from zero so that $S_X(f)$ is ∞. The signal is not of the power type either.

Problem [redacted] 7

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x(t-nT_s) &= x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \frac{1}{T_s} x(t) \star \sum_{n=-\infty}^{\infty} e^{j2\pi \frac{n}{T_s} t} \\
&= \frac{1}{T_s} \mathcal{F}^{-1}\left[X(f) \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_s}) \right] \\
&= \frac{1}{T_s} \mathcal{F}^{-1}\left[\sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) \delta(f - \frac{n}{T_s}) \right] \\
&= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) e^{j2\pi \frac{n}{T_s} t}
\end{aligned}$$

If we set $t = 0$ in the previous relation we obtain Poisson's sum formula

$$\sum_{n=-\infty}^{\infty} x(-nT_s) = \sum_{m=-\infty}^{\infty} x(mT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right)$$