ELG3175
Introduction to Communication Systems

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Solutions

Assignment #3

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Problem [redacted] 1

Let the response of the LTI system be $h(t)$ with Fourier transform $H(f)$. Then, from the convolution theorem we obtain

$$Y(f) = H(f)X(f) \Longrightarrow \Lambda(f) = \Pi(f)H(f)$$

However, this relation cannot hold since $\Pi(f) = 0$ for $\frac{1}{2} < |f|$ whereas $\Lambda(f) \neq 0$ for $1 < |f| \leq 1/2$.

Problem [redacted] 2

For no aliasing to occur we must sample at the Nyquist rate

$$f_s = 2 \cdot 6000 \text{ samples/sec} = 12000 \text{ samples/sec}$$

With a guard band of 2000

$$f_s - 2W = 2000 \Longrightarrow f_s = 14000$$

The reconstruction filter should not pick-up frequencies of the images of the spectrum $X(f)$. The nearest image spectrum is centered at f_s and occupies the frequency band $[f_s - W, f_s + W]$. Thus the highest frequency of the reconstruction filter ($= 10000$) should satisfy

$$10000 \leq f_s - W \Longrightarrow f_s \geq 16000$$

For the value $f_s = 16000$, K should be such that

$$K \cdot f_s = 1 \Longrightarrow K = (16000)^{-1}$$

Problem [redacted] 3

$$x(t) = A\text{sinc}(1000\pi t) \Longrightarrow X(f) = \frac{A}{1000}\Pi(\frac{f}{1000})$$

Thus the bandwidth W of $x(t)$ is $1000/2 = 500$. Since we sample at $f_s = 2000$ there is a gap between the image spectra equal to

$$2000 - 500 - W = 1000$$

The reconstruction filter should have a bandwidth W' such that $500 < W' < 1500$. A filter that satisfy these conditions is

$$H(f) = T_s\Pi\left(\frac{f}{2W'}\right) = \frac{1}{2000}\Pi\left(\frac{f}{2W'}\right)$$

and the more general reconstruction filters have the form

$$H(f) = \begin{cases} \frac{1}{2000} & |f| < 500 \\ \text{arbitrary} & 500 < |f| < 1500 \\ 0 & |f| > 1500 \end{cases}$$

Problem [redacted] 4

1) $W = 50$Hz so that $T_s = 1/2W = 10^{-2}$sec. The reconstructed signal is

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$$

$$\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} x(nT_s)\operatorname{sinc}\left(\frac{t}{T_s} - n\right) \\
&= -\sum_{n=-4}^{-1} \operatorname{sinc}\left(\frac{t}{T_s} - n\right) + \sum_{n=1}^{4} \operatorname{sinc}\left(\frac{t}{T_s} - n\right)
\end{aligned}$$

With $T_s = 10^{-2}$ and $t = 5 \cdot 10^{-3}$ we obtain

$$\begin{aligned}
x(.005) &= -\sum_{n=1}^{4} \operatorname{sinc}\left(\frac{1}{2} + n\right) + \sum_{n=1}^{4} \operatorname{sinc}\left(\frac{1}{2} - n\right) \\
&= -[\operatorname{sinc}(\frac{3}{2}) + \operatorname{sinc}(\frac{5}{2}) + \operatorname{sinc}(\frac{7}{2}) + \operatorname{sinc}(\frac{9}{2})] \\
&\quad +[\operatorname{sinc}(-\frac{1}{2}) + \operatorname{sinc}(-\frac{3}{2}) + \operatorname{sinc}(-\frac{5}{2}) + \operatorname{sinc}(-\frac{7}{2})] \\
&= \operatorname{sinc}(\frac{1}{2}) - \operatorname{sinc}(\frac{9}{2}) = \frac{2}{\pi}\sin(\frac{\pi}{2}) - \frac{2}{9\pi}\sin(\frac{9\pi}{2}) \\
&= \frac{16}{9\pi}
\end{aligned}$$

where we have used the fact that $\operatorname{sinc}(t)$ is an even function.

This signal is limited in time $\Rightarrow$ it's a power-type

2) Note that (see Problem 2.41)

$$\int_{-\infty}^{\infty} \operatorname{sinc}(2Wt - m)\operatorname{sinc}^*(2Wt - n)dt = \frac{1}{2W}\delta_{mn}$$

with δ_{mn} the Kronecker delta. Thus,

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)x^*(t)dt \\
&= \sum_{n=-\infty}^{\infty} x(nT_s)x^*(mT_s)\int_{-\infty}^{\infty} \operatorname{sinc}(2Wt - m)\operatorname{sinc}^*(2Wt - n)dt \\
&= \sum_{n=-\infty}^{\infty} |x(nT_s)|^2 \frac{1}{2W}
\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2W}\left[\sum_{n=-4}^{-1} 1 + \sum_{n=1}^{4} 1\right] = \frac{4}{W} = 8 \cdot 10^{-2}$$

#5

$$x(t) = e^{-400\pi t} u(t)$$

$$S_x(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt = \frac{1}{-(j2\pi f + 400\pi)};$$

$$E_x(t) = \int_0^{\infty} e^{-800\pi t} dt = \frac{1}{800\pi};$$

$$x(t) \longrightarrow \boxed{F} \longrightarrow y(t) \qquad E_y = E_x/2 = \frac{1}{1600\pi};$$

$$E_y = 2\int_0^B |S_x(f)|^2 df = 2\int_0^B \frac{1}{(400\pi)^2 + (2\pi f)^2} df =$$

$$= \frac{2}{(400\pi)^2} \int_0^B \frac{df}{1 + \left(\frac{f}{200}\right)^2} = \frac{2 \cdot 200}{(400\pi)^2} \tan^{-1}\left(\frac{B}{200}\right) =$$

$$= \frac{1}{400\pi^2} \tan^{-1}\left(\frac{B}{200}\right) = \frac{1}{1600\pi} \Rightarrow$$

$$\Rightarrow \tan^{-1}\left(\frac{B}{200}\right) = \pi/4 \Rightarrow B = 200 \text{ Hz}.$$

#6 The maximum frequency in the Spectrum of $x(t)$ is $f_{max} = \omega_0/2$, and $S_x(f) = 0, |f| > f_{max}$

(a) Since $S_{\frac{d^2x}{dt^2}}(f) = -\omega^2 S_x(f)$, it is also limited by f_{max}. Hence, the Nyquist rate is ω_0.

(b) The max. frequency in $x^2(t-5)$ is $2f_{max}$ (convolution of spectra), hence, using multiplication property, we conclude that max. frequency of $x(t)\,x^2(t-5)$ is $3f_{max}$. (Convolution of spectra). The ~~samp~~ Nyquist rate $\omega_s =$ $= 6\omega_m = 3\omega_0$

(c) Since convolution of signals results in multiplication of their spectra, max frequency in $x(t) * x^2(t-5)$ is f_{max}. Hence, $\omega_s = \omega_0$.

(d) The spectrum of $\int_{-\infty}^{t} x(t)\,dt$ is $\leftrightarrow \frac{S_x(f)}{j2\pi f} + \frac{1}{2} S_x(0)\delta(f)$

Hence, its max. frequency is f_{max}. Hence, $\omega_s = \omega_0$.

Problem [redacted] 7

1) No. The input $\Pi(t)$ has a spectrum with zeros at frequencies $f = k$, $(k \neq 0,\ k \in \mathcal{Z})$ and the information about the spectrum of the system at those frequencies will not be present at the output. The spectrum of the signal $\cos(2\pi t)$ consists of two impulses at $f = \pm 1$ but we do not know the response of the system at these frequencies.

2)

$$
\begin{aligned}
h_1(t) \star \Pi(t) &= \Pi(t) \star \Pi(t) = \Lambda(t) \\
h_2(t) \star \Pi(t) &= (\Pi(t) + \cos(2\pi t)) \star \Pi(t) \\
&= \Lambda(t) + \frac{1}{2}\mathcal{F}^{-1}\left[\delta(f-1)\text{sinc}^2(f) + \delta(f+1)\text{sinc}^2(f)\right] \\
&= \Lambda(t) + \frac{1}{2}\mathcal{F}^{-1}\left[\delta(f-1)\text{sinc}^2(1) + \delta(f+1)\text{sinc}^2(-1)\right] \\
&= \Lambda(t)
\end{aligned}
$$

Thus both signals are candidates for the impulse response of the system.

3) $\mathcal{F}[u_{-1}(t)] = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$. Thus the system has a nonzero spectrum for every f and all the frequencies of the system will be excited by this input. $\mathcal{F}[e^{-at}u_{-1}(t)] = \frac{1}{a+j2\pi f}$. Again the spectrum is nonzero for all f and the response to this signal uniquely determines the system. In general the spectrum of the input must not vanish at any frequency. In this case the influence of the system will be present at the output for every frequency.

#8

when the signal is passed through a low-pass filter, we are in effect performing band-limited interpolation. this results in the signal:

$$y(t) = \cos\left(\frac{\omega_s}{2}t\right)\cos(\varphi)$$

7.39. **(a)** Using Trigonometric identities,

$$\cos\left(\frac{\omega_s}{2}t+\phi\right) = \cos\left(\frac{\omega_s}{2}t\right)\cos(\phi) - \sin\left(\frac{\omega_s}{2}t\right)\sin(\phi).$$

Therefore,

$$g(t) = -\sin\left(\frac{\omega_s}{2}t\right)\sin(\phi).$$

(b) By replacing ω_s with $2\pi/T$, and t by NT in the above equation, we get

$$\begin{aligned} g(nT) &= -\sin\left(\frac{2\pi}{2T}nT\right)\sin(\phi) \\ &= -\sin(n\pi)\sin(\phi). \end{aligned}$$

Clearly, the right-hand side of the above equation is zero for $n = 0, \pm 1, \pm 2, \cdots$.

(c) From parts (a) and (b), we get

$$\begin{aligned} x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) &= \sum_{n=-\infty}^{\infty} \delta(t-nT)\left\{\cos\left(\frac{\omega_s}{2}nT\right)\cos(\phi) + g(nT)\right\} \\ &= \sum_{n=-\infty}^{\infty} \delta(t-nT)\cos\left(\frac{\omega_s}{2}nT\right)\cos(\phi) \end{aligned}$$

4

Problem [redacted] **9**

1)

$$x_1(t) = \sum_{n=-\infty}^{\infty} (-1)^n x(nT_s)\delta(t-nT_s) = x(t)\sum_{n=-\infty}^{\infty}(-1)^n\delta(t-nT_s)$$

$$= x(t)\left[\sum_{l=-\infty}^{\infty}\delta(t-2lT_s) - \sum_{l=-\infty}^{\infty}\delta(t-T_s-2lT_s)\right]$$

Thus

$$X_1(f) = X(f)\star\left[\frac{1}{2T_s}\sum_{l=-\infty}^{\infty}\delta(f-\frac{l}{2T_s}) - \frac{1}{2T_s}\sum_{l=-\infty}^{\infty}\delta(f-\frac{l}{2T_s})e^{-j2\pi fT_s}\right]$$

$$= \frac{1}{2T_s}\sum_{l=-\infty}^{\infty}X(f-\frac{l}{2T_s}) - \frac{1}{2T_s}\sum_{l=-\infty}^{\infty}X(f-\frac{l}{2T_s})e^{-j2\pi\frac{l}{2T_s}T_s}$$

$$= \frac{1}{2T_s}\sum_{l=-\infty}^{\infty}X(f-\frac{l}{2T_s}) - \frac{1}{2T_s}\sum_{l=-\infty}^{\infty}X(f-\frac{l}{2T_s})(-1)^l$$

$$= \frac{1}{T_s}\sum_{l=-\infty}^{\infty}X(f-\frac{1}{2T_s}-\frac{l}{T_s})$$

2) The spectrum of $x(t)$ occupies the frequency band $[-W, W]$. Suppose that from the periodic spectrum $X_1(f)$ we isolate $X_k(f) = \frac{1}{T_s}X(f-\frac{1}{2T_s}-\frac{k}{T_s})$, with a bandpass filter, and we use it to reconstruct $x(t)$. Since $X_k(f)$ occupies the frequency band $[2kW, 2(k+1)W]$, then for all k, $X_k(f)$ cannot cover the whole interval $[-W, W]$. Thus at the output of the reconstruction filter there will exist frequency components which are not present in the input spectrum. Hence, the reconstruction filter has to be a time-varying filter. To see this in the time domain, note that the original spectrum has been shifted by $f' = \frac{1}{2T_s}$. In order to bring the spectrum back to the origin and reconstruct $x(t)$ the sampled signal $x_1(t)$ has to be multiplied by $e^{-j2\pi\frac{1}{2T_s}t} = e^{-j2\pi Wt}$. However the system described by

$$y(t) = e^{j2\pi Wt}x(t)$$

is a time-varying system.

3) Using a time-varying system we can reconstruct $x(t)$ as follows. Use the bandpass filter $T_s\Pi(\frac{f-W}{2W})$ to extract the component $X(f-\frac{1}{2T_s})$. Invert $X(f-\frac{1}{2T_s})$ and multiply the resultant signal by $e^{-j2\pi Wt}$. Thus

$$x(t) = e^{-j2\pi Wt}\mathcal{F}^{-1}\left[T_s\Pi(\frac{f-W}{2W})X_1(f)\right]$$