ELG3175
Introduction to Communication Systems

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Solutions

Assignment #4

Problem [redacted] 7

1)

$$
\begin{aligned}
u(t) &= 5\cos(1800\pi t) + 20\cos(2000\pi t) + 5\cos(2200\pi t) \\
&= 20\left(1 + \frac{1}{2}\cos(200\pi t)\right)\cos(2000\pi t)
\end{aligned}
$$

The modulating signal is $m(t) = \cos(2\pi 100t)$ whereas the carrier signal is $c(t) = 20\cos(2\pi 1000t)$.

2) Since $-1 \leq \cos(2\pi 100t) \leq 1$, we immediately have that the modulation index is $\alpha = \frac{1}{2}$.

3) The power of the carrier component is $P_{\text{carrier}} = \frac{400}{2} = 200$, whereas the power in the sidebands is $P_{\text{sidebands}} = \frac{400\alpha^2}{2} = 50$. Hence,

$$
\frac{P_{\text{sidebands}}}{P_{\text{carrier}}} = \frac{50}{200} = \frac{1}{4}
$$

Problem [redacted] 4

The modulated signal is

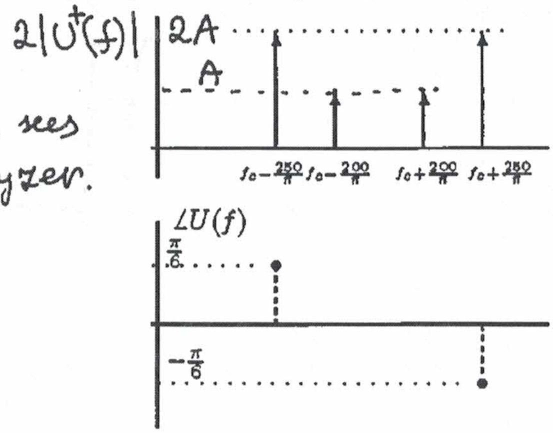
$$
\begin{aligned}
u(t) &= m(t)c(t) = Am(t)\cos(2\pi 4\times 10^3 t)\\
&= A\left[2\cos(2\pi\frac{200}{\pi}t) + 4\sin(2\pi\frac{250}{\pi}t+\frac{\pi}{3})\right]\cos(2\pi 4\times 10^3 t)\\
&= A\cos(2\pi(4\times 10^3+\frac{200}{\pi})t) + A\cos(2\pi(4\times 10^3-\frac{200}{\pi})t)\\
&\quad +2A\sin(2\pi(4\times 10^3+\frac{250}{\pi})t+\frac{\pi}{3}) - 2A\sin(2\pi(4\times 10^3-\frac{250}{\pi})t-\frac{\pi}{3})
\end{aligned}
$$

Taking the Fourier transform of the previous relation, we obtain

$$
\begin{aligned}
U(f) &= A\left[\delta(f-\frac{200}{\pi}) + \delta(f+\frac{200}{\pi}) + \frac{2}{j}e^{j\frac{\pi}{3}}\delta(f-\frac{250}{\pi}) - \frac{2}{j}e^{-j\frac{\pi}{3}}\delta(f+\frac{250}{\pi})\right]\\
&\quad \star\frac{1}{2}[\delta(f-4\times 10^3)+\delta(f+4\times 10^3)]\\
&= \frac{A}{2}\left[\delta(f-4\times 10^3-\frac{200}{\pi}) + \delta(f-4\times 10^3+\frac{200}{\pi})\right.\\
&\quad +2e^{-j\frac{\pi}{6}}\delta(f-4\times 10^3-\frac{250}{\pi}) + 2e^{j\frac{\pi}{6}}\delta(f-4\times 10^3+\frac{250}{\pi})\\
&\quad +\delta(f+4\times 10^3-\frac{200}{\pi}) + \delta(f+4\times 10^3+\frac{200}{\pi})\\
&\quad \left.+2e^{-j\frac{\pi}{6}}\delta(f+4\times 10^3-\frac{250}{\pi}) + 2e^{j\frac{\pi}{6}}\delta(f+4\times 10^3+\frac{250}{\pi})\right]
\end{aligned}
$$

The next figure depicts the magnitude and the phase of the spectrum as seen on spectrum analyzer

this is what one sees on a spectrum analyzer.



To find the power content of the modulated signal we write $u^2(t)$ as

$$
\begin{aligned}
u^2(t) &= A^2\cos^2(2\pi(4\times 10^3+\frac{200}{\pi})t) + A^2\cos^2(2\pi(4\times 10^3-\frac{200}{\pi})t)\\
&\quad +4A^2\sin^2(2\pi(4\times 10^3+\frac{250}{\pi})t+\frac{\pi}{3}) + 4A^2\sin^2(2\pi(4\times 10^3-\frac{250}{\pi})t-\frac{\pi}{3})\\
&\quad +\text{terms of cosine and sine functions in the first power}
\end{aligned}
$$

Hence,

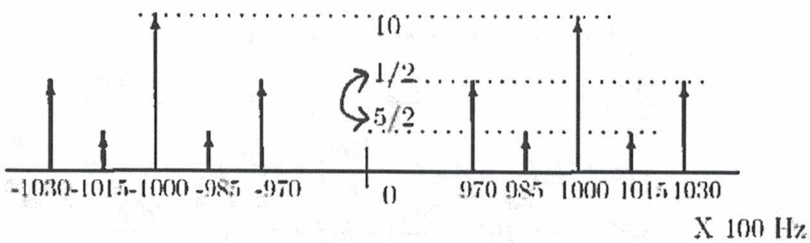
$$P = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} u^2(t)dt = \frac{A^2}{2}+\frac{A^2}{2}+\frac{4A^2}{2}+\frac{4A^2}{2} = 5A^2$$

Problem [redacted] 6

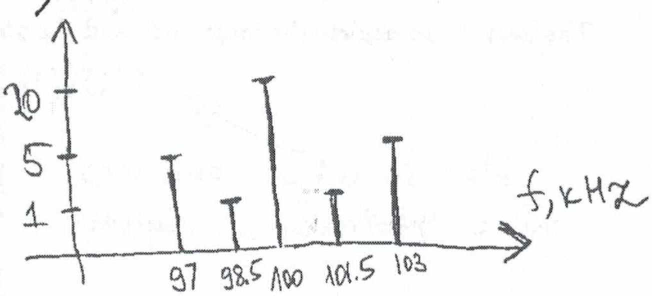
1) The spectrum of $u(t)$ is

$$
\begin{aligned}
U(f) &= \frac{20}{2}\left[\delta(f-f_c)+\delta(f+f_c)\right] \\
&+\frac{2}{4}\left[\delta(f-f_c-1500)+\delta(f-f_c+1500)\right. \\
&\left.+\delta(f+f_c-1500)+\delta(f+f_c+1500)\right] \\
&+\frac{10}{4}\left[\delta(f-f_c-3000)+\delta(f-f_c+3000)\right. \\
&\left.+\delta(f+f_c-3000)+\delta(f+f_c+3000)\right]
\end{aligned}
$$

The next figure depicts the spectrum of $u(t)$.



On a spectrum analyzer, one sees:



2) The square of the modulated signal is

$$
\begin{aligned}
u^2(t) &= 400\cos^2(2\pi f_c t)+\cos^2(2\pi(f_c-1500)t)+\cos^2(2\pi(f_c+1500)t) \\
&+25\cos^2(2\pi(f_c-3000)t)+25\cos^2(2\pi(f_c+3000)t) \\
&+\text{ terms that are multiples of cosines}
\end{aligned}
$$

#6

If we integrate $u^2(t)$ from $-\frac{T}{2}$ to $\frac{T}{2}$, normalize the integral by $\frac{1}{T}$ and take the limit as $T \to \infty$, then all the terms involving cosines tend to zero, whereas the squares of the cosines give a value of $\frac{1}{2}$. Hence, the power content at the frequency $f_c = 10^5$ Hz is $P_{f_c} = \frac{400}{2} = 200$, the power content at the frequency P_{f_c+1500} is the same as the power content at the frequency P_{f_c-1500} and equal to $\frac{1}{2}$, whereas $P_{f_c+3000} = P_{f_c-3000} = \frac{25}{2}$.

3)

$$\begin{aligned} u(t) &= (20 + 2\cos(2\pi 1500t) + 10\cos(2\pi 3000t))\cos(2\pi f_c t) \\ &= 20(1 + \frac{1}{10}\cos(2\pi 1500t) + \frac{1}{2}\cos(2\pi 3000t))\cos(2\pi f_c t) \end{aligned}$$

This is the form of a conventional AM signal with message signal

$$\begin{aligned} m(t) &= \frac{1}{10}\cos(2\pi 1500t) + \frac{1}{2}\cos(2\pi 3000t) \\ &= \cos^2(2\pi 1500t) + \frac{1}{10}\cos(2\pi 1500t) - \frac{1}{2} \end{aligned}$$

$\min[m(t)] = -\frac{1}{10} - \frac{1}{2}$

$\max[m(t)] = \frac{1}{10} + \frac{1}{2} = \frac{3}{5}$;

$\approx -1/2$

The minimum of $g(z) = z^2 + \frac{1}{10}z - \frac{1}{2}$ is achieved for $z = -\frac{1}{20}$ and it is $\min(g(z)) = -\frac{201}{400}$. Since $z = -\frac{1}{20}$ is in the range of $\cos(2\pi 1500t)$, we conclude that the minimum value of $m(t)$ is $-\frac{201}{400}$. $\approx -\frac{1}{2}$

Hence, the modulation index is

$$a = \cancel{\frac{201}{400}} \quad \frac{\max - \min}{2} = \frac{\frac{3}{5} + \frac{1}{2}}{2} = \frac{11}{20} \approx \frac{1}{2}$$

4)

$$\begin{aligned} u(t) &= 20\cos(2\pi f_c t) + \cos(2\pi(f_c - 1500)t) + \cos(2\pi(f_c - 1500)t) \\ &+ 5\cos(2\pi(f_c - 3000)t) + 5\cos(2\pi(f_c + 3000)t) \end{aligned}$$

The power in the sidebands is

$$P_{\text{sidebands}} = \frac{1}{2} + \frac{1}{2} + \frac{25}{2} + \frac{25}{2} = 26$$

The total power is $P_{\text{total}} = P_{\text{carrier}} + P_{\text{sidebands}} = 200 + 26 = 226$. The ratio of the sidebands power to the total power is

$$\frac{P_{\text{sidebands}}}{P_{\text{total}}} = \frac{26}{226} \approx \frac{1}{10}$$

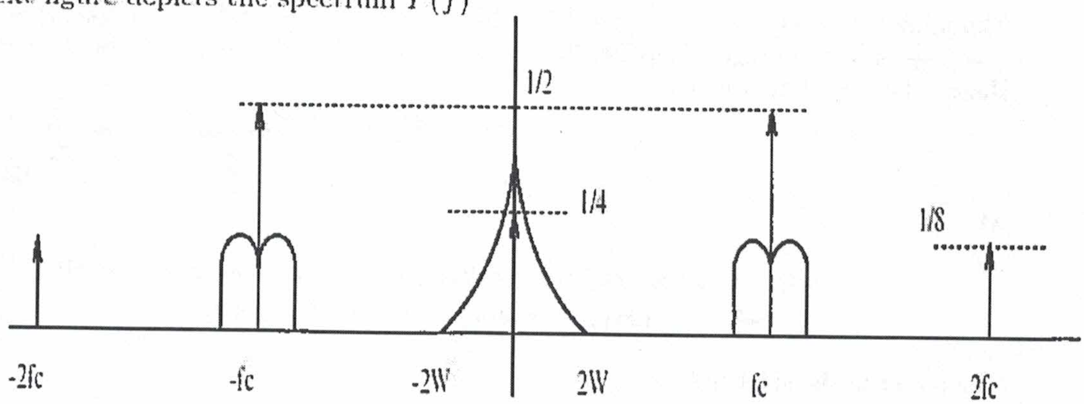
Problem [redacted] #5

$$\begin{aligned}
y(t) &= x(t) + \frac{1}{2}x^2(t) \\
&= m(t) + \cos(2\pi f_c t) + \frac{1}{2}\left(m^2(t) + \cos^2(2\pi f_c t) + 2m(t)\cos(2\pi f_c t)\right) \\
&= m(t) + \cos(2\pi f_c t) + \frac{1}{2}m^2(t) + \frac{1}{4} + \frac{1}{4}\cos(2\pi 2 f_c t) + m(t)\cos(2\pi f_c t)
\end{aligned}$$

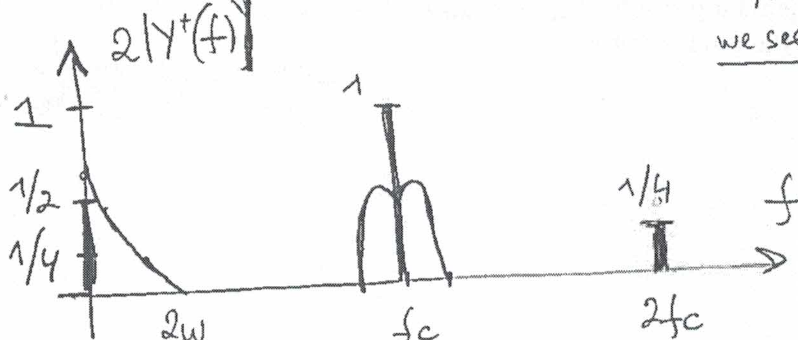
Taking the Fourier transform of the previous, we obtain

$$\begin{aligned}
Y(f) &= M(f) + \frac{1}{2}M(f) \star M(f) + \frac{1}{2}\left(M(f - f_c) + M(f + f_c)\right) \\
&\quad + \frac{1}{4}\delta(f) + \frac{1}{2}\left(\delta(f - f_c) + \delta(f + f_c)\right) + \frac{1}{8}\left(\delta(f - 2f_c) + \delta(f + 2f_c)\right)
\end{aligned}$$

The next figure depicts the spectrum $Y(f)$



One the spectrum analyzer, we see $2|Y^+(f)|$ → except for $f=0$: we see $Y(0)$!

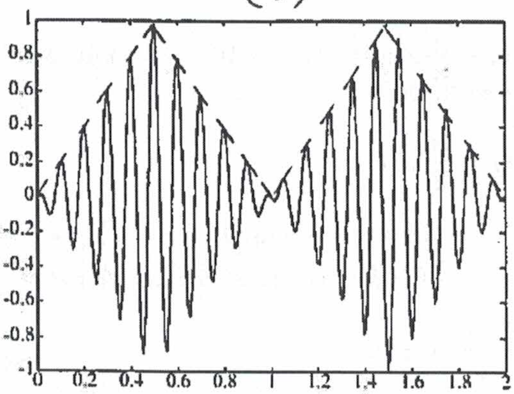


All "heights" of delta-functions have a finite height on a spectrum analyzer, equal to the magnitude of appropriate sinusoid in the spectrum.

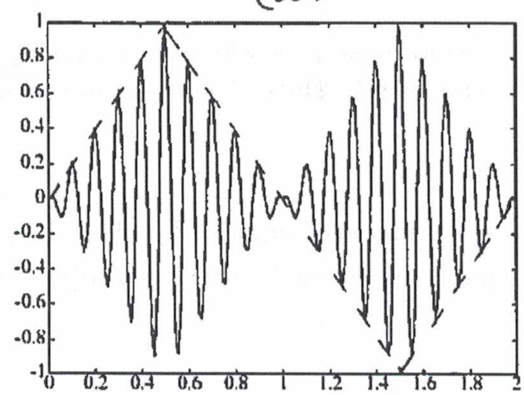
Problem [redacted] **2**

The following figure shows the modulated signals for $A = 1$ and $f_0 = 10$. As it is observed both signals have the same envelope but there is a phase reversal at $t = 1$ for the second signal $Am_2(t)\cos(2\pi f_0 t)$ ([redacted] Left plot). This discontinuity is shown clearly in the next figure where we plotted $Am_2(t)\cos(2\pi f_0 t)$ with $f_0 = 3$.

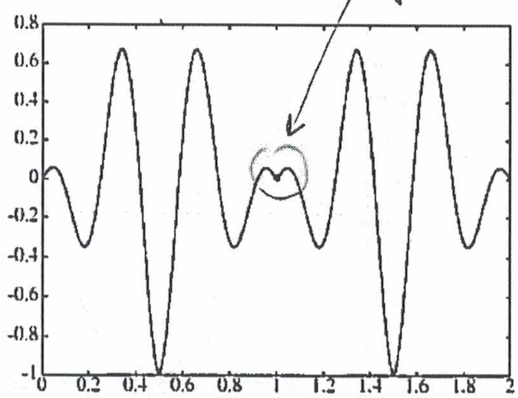
(b)



(a)



(a) → expanded!



Problem [redacted] 4 5

The mixed signal $y(t)$ is given by

$$\begin{aligned} y(t) &= u(t) \cdot x_L(t) = Am(t)\cos(2\pi f_c t)\cos(2\pi f_c t + \theta) \\ &= \frac{A}{2} m(t)\left[\cos(2\pi 2 f_c t + \theta) + \cos(\theta)\right] \end{aligned}$$

The lowpass filter will cut-off the frequencies above W, where W is the bandwidth of the message signal $m(t)$. Thus, the output of the lowpass filter is

$$z(t) = \frac{A}{2} m(t)\cos(\theta)$$

If the power of $m(t)$ is P_M, then the power of the output signal $z(t)$ is $P_{\text{out}} = P_M \frac{A^2}{4}\cos^2(\theta)$. The power of the modulated signal $u(t) = Am(t)\cos(2\pi f_c t)$ is $P_U = \frac{A^2}{2} P_M$. Hence,

$$\frac{P_{\text{out}}}{P_U} = \frac{1}{2}\cos^2(\theta)$$

A plot of $\frac{P_{\text{out}}}{P_U}$ for $0 \le \theta \le \pi$ is given in the next figure.