Université d'Ottawa
Faculté de génie

École de science informatique
et de génie électrique

uOttawa

L'Université canadienne
Canada's university

University of Ottawa
Faculty of Engineering

School of Electrical Engineering
and Computer Science

ELG 3126 **RANDOM SIGNALS AND SYSTEMS** **Winter 2023**

ASSIGNMENT 1

Set Theory

(due at 8:30 AM Thursday, Jan. 19, submission in class)

1. Your University of Ottawa student number has k distinct digits in it. State the set of these digits and all the subsets of this set that have two or fewer elements

2. If $\mathcal{S} = \{1, 3, 5, 7, 9, 11\}$, $\mathcal{A} = \{3, 5, 7\}$ and $\mathcal{B} = \{3, 5, 9\}$, determine (i) $\overline{\mathcal{A}}$, (ii) $\mathcal{A} \cap \mathcal{B}$, (iii) $\mathcal{B} - \mathcal{A}$, (iv) $\mathcal{A} - \mathcal{B}$, and (v) $\overline{\mathcal{A}} \cap \mathcal{B}$.

3. Use De Morgan's laws to show that (i) $\overline{\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C})} = (\overline{\mathcal{A}} \cap \overline{\mathcal{B}}) \cup (\overline{\mathcal{A}} \cap \overline{\mathcal{C}})$, (ii) $\overline{\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}} = \overline{\mathcal{A}} \cup \overline{\mathcal{B}} \cup \overline{\mathcal{C}}$.

4. Prove that a finite set with N elements has exactly 2^N subsets.

5. Prove for arbitrary sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ that if $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{C}$, then $\mathcal{A} \subset \mathcal{C}$.

6. Determine whether the following sets are finite, infinitely countable or uncountable, and their cardinality:
   $\mathcal{A} = \{\varnothing, \text{red}, \text{green}, 0, 1\}$,
   $\mathcal{B} = \{x \mid x^2 + 2x + 1 = 0, x \in \mathbb{R}\}$,
   $\mathcal{C} = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$,
   $\mathcal{D} = \{$students currently attending Lisgar Collegiate$\}$,
   $\mathcal{E} = \{$female students currently attending Lisgar Collegiate$\}$,
   $\mathcal{F} = \{$female students currently enrolled in an engineering program at the University of Ottawa and who were born after Jan. 1, 2010$\}$,
   $\mathcal{G} = \{x \mid x \in \mathbb{R}, 0 \leq x < 3\} = [0, 3)$,
   $\mathcal{H} = \{x \mid x^2 - 3x + 2 = 0, x \in \mathbb{R}\}$,
   $\mathcal{I} = \{-1, 2\} \cup [1, 3]$.
   Determine if any of these sets is equal to any other of these sets and which is a subset of another.

7. Suppose that, for $i = 1, 2, 3, \ldots$, the set $\mathcal{A}_i$ is the set of real numbers from $i/(2[i+1]^2)$ to $3-(i/[i+1])$ inclusive (i.e., $\mathcal{A}_i = [i/(2[i+1]^2), 3 - (i/[i+1])]$). What is the (countable) union of these sets?

8. Suppose that, for $i = 1, 2, 3, \ldots$, the set $\mathcal{A}_i$ is the set of real numbers from $1/[i^2 + 2]$ to $2 + [2/i^2]$ exclusive (i.e., $\mathcal{A}_i = (1/[i + 2], 2 + [2/i^2])$). What is the (countable) intersection of these sets?

9. Suppose that, for $i = 1, 2, 3, \ldots$, the set $\mathcal{A}_i$ is the set of real numbers from e^{-i^2} to $3 + (1/i)$ inclusive (i.e., $\mathcal{A}_i = [e^{-i^2}, 3 + (1/i)]$). What is the (countable) union of these sets?

10. Prove that union of any finite number of countably infinite sets is countably infinite.
    Hint: Recall that a set is countable if it is finite or countably infinite.
    Remark: It is not to hard to prove an even stronger statement: *Any countable union of countable sets is countable.*

11. If $\mathcal{A}$ and $\mathcal{B}$ are two countably infinite sets, what are the possible cardinalities of (i) $\mathcal{A} \cup \mathcal{B}$, (ii) $\mathcal{A} \cap \mathcal{B}$ and (iii) $\mathcal{A} - \mathcal{B}$? Give examples of the possibilities.

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ASSIGNMENT 1

SOLUTIONS

1/ The answer depends on the student number. For the fictitious number 9103219, the set of digits that appear is $\{0,1,2,3,9\}$, and so the subsets of 0, 1 or 2 elements are the 16 sets $\{\},\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{9\}$, $\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{0,9\}$, $\{1,2\}$, $\{1,3\}$, $\{1,9\}$, $\{2,3\}$, $\{2,9\}$, $\{3,9\}$. (In general, for a set of n objects there are $\binom{n}{k}$ subsets of size k, so there is 1 subset of size 0, n subsets of size 1 and $\frac{1}{2}n(n-1)$ subsets of size 2.)

2/ (i) $\overline{\mathcal{A}} = \{1,9,11\}$, (ii) $\mathcal{A}\cap\mathcal{B} = \{3,5\}$, (iii) $\mathcal{B}-\mathcal{A} = \{9\}$, (iv) $\mathcal{A}-\mathcal{B} = \{7\}$, and (v) $\overline{\mathcal{A}}\cap\mathcal{B} = \{9\}$.

3/ (i) By De Morgan's rule

$$\begin{aligned}\overline{\mathcal{A}\cup(\mathcal{B}\cap\mathcal{C})} &= \overline{\mathcal{A}}\cap\overline{(\mathcal{B}\cap\mathcal{C})} = \overline{\mathcal{A}}\cap(\overline{\mathcal{B}}\cup\overline{\mathcal{C}})\\ &= (\overline{\mathcal{A}}\cap\overline{\mathcal{B}})\cup(\overline{\mathcal{A}}\cap\overline{\mathcal{C}}) \quad \text{(since intersection distributes over union)}\end{aligned}$$

(ii)

$$\begin{aligned}\overline{\mathcal{A}\cap\mathcal{B}\cap\mathcal{C}} &= \overline{\mathcal{A}\cap(\mathcal{B}\cap\mathcal{C})} = \overline{\mathcal{A}}\cup\overline{(\mathcal{B}\cap\mathcal{C})}\\ &= \overline{\mathcal{A}}\cup(\overline{\mathcal{B}}\cup\overline{\mathcal{C}}) = \overline{\mathcal{A}}\cup\overline{\mathcal{B}}\cup\overline{\mathcal{C}}.\end{aligned}$$

4/ If we associate each of the N elements with a digit in an N-bit binary number, then each subset of the set can be identified with an N-bit binary number by making the ith digit a 1 if the element is included in the subset and a 0 otherwise. The number of subsets is then the same as the number of N-digit binary numbers—of which there are 2^N (representing values 0 to 2^N-1).

5/ Assume that $a\in\mathcal{A}$. Then since $\mathcal{A}\subset\mathcal{B}$, we have $a\in\mathcal{B}$. Since $\mathcal{B}\subset\mathcal{C}$ and we have that if $a\in\mathcal{B}$, then $a\in\mathcal{C}$. Thus if $a\in\mathcal{A}$, then $a\in\mathcal{C}$; so we conclude that $\mathcal{A}\subset\mathcal{C}$.

6/ $\mathcal{A}$ is a finite set of five elements and so has cardinality 5.

$\mathcal{B}=\{-1\}$ (since $x^2+2x+1=0$ has but one solution), so $\mathcal{B}$ is a finite set of one element and so has cardinality 1,

$\mathcal{C}$ is a countably infinite set and so has cardinality $\aleph_0$,

$\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ are obviously finite sets (with $\mathcal{F}=\{\}$ as there are no students enrolled in an engineering program aged 13 or less), with $\mathcal{F}$ having cardinality 0 and the cardinality of $\mathcal{D}$ and $\mathcal{F}$ depends the registration numbers for Lisgar (which are not provided),

$\mathcal{G}$ is an uncountable set with the same cardinality as the entire set of real numbers $\mathbb{R}$ which is $\aleph_1$ [we can see this by noting there is a 1-1 mapping between (0,3) and $\mathbb{R}$ in the form of the invertible mapping $y=\tan(\pi[x-\frac{3}{2}]/3)$], which tells us that $(0,3)$ has the same cardinality as $\mathbb{R}$, which is $\aleph_1$. But $(0,3)$ is a subset of $[0.3)$ so the cardinality of $[0.3)$ must be greater than or equal to the cardinality of $(0,3)$, $\aleph_1$. But $[0,3)$ is a subset of $\mathbb{R}$ so the cardinality must be less that or equal to the cardinality of $\mathbb{R}$. These two facts give us that $[0,3)$ must have cardinaility $\aleph_1$.

$\mathcal{H}=\{1,2\}$ and so is a finite set with cardinality 2,

$\mathcal{I}$ is an uncountable set with cardinality $\aleph_1$.

Since $\mathcal{F}=\varnothing$, it is a subset of every other set. Otherwise, $\mathcal{B}\subset\mathcal{I}$, $\mathcal{E}\subset\mathcal{D}$, $\mathcal{G}\subset\mathcal{I}$, $\mathcal{H}\subset\mathcal{C}$, $\mathcal{H}\subset\mathcal{G}$ and $\mathcal{H}\subset\mathcal{I}$.

7/ $\mathcal{A}_1=[\frac{1}{8},3-\frac{1}{2}]$, $\mathcal{A}_2=[\frac{2}{18},3-\frac{2}{3}]$, $\mathcal{A}_3=[\frac{3}{32},3-\frac{3}{4}]$, etc., so $\bigcup_{i=1}^{\infty}\mathcal{A}_i=(0,2\frac{1}{2}]$.

8/ $\mathcal{A}_1=(\frac{1}{2},3)$, $\mathcal{A}_2=(\frac{1}{3},2\frac{1}{2})$, $\mathcal{A}_3=(\frac{1}{4},2\frac{1}{3})$, $\mathcal{A}_4=(\frac{1}{5},2\frac{1}{4})$, etc., so $\bigcap_{i=1}^{\infty}\mathcal{A}_i=(\frac{1}{2},2]$.

9/ $\mathcal{A}_1 = [e^{-1}, 4]$, $\mathcal{A}_2 = [2e^{-4}, 3\frac{1}{2}]$, $\mathcal{A}_3 = [3e^{-9}, 3\frac{1}{3}]$, $\mathcal{A}_4 = [4e^{-16}, 3\frac{1}{4}]$, etc., so $\bigcup_{i=1}^{\infty} \mathcal{A}_i = (0, 4]$.

10/ We can establish the result that the slightly stronger statement that the finite union of any countable sets is countable, from which the claim in the problem follows as a special case.

Proof: It is obvious that the union of a finite number of finite sets is finite. Also, we can easily see that the union of a finite set with a countably infinite set is countably infinite: remove from the finite set any elements in the infinite set, then just list the remaining elements followed by the elements of the infinite set counting as we go, giving us a one to one correspondence with the elements in the union and the natural numbers.

The union of two countably infinite sets $\mathcal{A} = \{a_1, a_2, a_3, \ldots\}$ and $\mathcal{B} = \{b_1, b_2, b_3, \ldots\}$ must also be countably infinite: First remove from $\mathcal{A}$ any elements in common with $\mathcal{B}$ giving the set $\hat{\mathcal{A}} = \mathcal{A} - \mathcal{B}$. Obviously $\mathcal{A} \cup \mathcal{B} = \hat{\mathcal{A}} \cup \mathcal{B}$. $\hat{\mathcal{A}}$ can only be a finite or countably infinite set. If $\hat{\mathcal{A}}$ is a finite set, then the union of $\mathcal{A}$ with $\mathcal{B}$ is the union of a finite set with an infinite set which the above has already argued is a countably infinite set. If $\hat{\mathcal{A}}$ is an infinite set $\{\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots\}$, the the union of $\mathcal{A}$ and $\mathcal{B}$ can be listed as $\{\hat{a}_1, b_1, \hat{a}_2, b_2, \hat{a}_3, b_3, \ldots\}$ and so the union is countably infinite.

If we are given a finite collection of countable sets, then their union is found by performing binary unions ($m - 1$ unions for m sets), each of which is of a type mentioned above and which always yields a countable set. Thus the final result must be countable.

11/ (i) The union of two countably infinite sets must always be countably infinite per the analysis in the preceding question. (Obviously the cardinality of a union of two sets is always greater than or equal to the cardinality of each of the sets, so it can't be finite.) For example:
(a) $\mathcal{A} = \mathbb{N}$, $\mathcal{B} = \{0, -1, -2, -3, \ldots\}$, $\mathcal{A} \cup \mathcal{B} = \mathbb{Z}$;
(b) $\mathcal{A} = \{0, 1, 2, 3, \ldots\}$, $\mathcal{B} = \{0, -1, -2, -3, \ldots\}$, $\mathcal{A} \cup \mathcal{B} = \mathbb{Z}$;
(c) $\mathcal{A} = \{0, 1, 2, 3, \ldots\}$, $\mathcal{B} = \{0, 2, 4, 6, \ldots\}$, $\mathcal{A} \cup \mathcal{B} = \{0, 1, 2, 3, \ldots\}$;
(d) $\mathcal{A} = \{0, 1, 2, 3, \ldots\}$, $\mathcal{B} = \{1, 2, 3, \ldots\}$, $\mathcal{A} \cup \mathcal{B} = \{0, 1, 2, 3, \ldots\}$;

(ii) the intersection of two countably infinite sets must be a countably set, which is to say it could be finite or countably infinite. For example:
(a) $\mathcal{A} = \mathbb{N}$, $\mathcal{B} = \{0, -1, -2, -3, \ldots\}$, $\mathcal{A} \cap \mathcal{B} = \varnothing$;
(b) $\mathcal{A} = \{0, 1, 2, 3, \ldots\}$, $\mathcal{B} = \{0, -1, -2, -3, \ldots\}$, $\mathcal{A} \cap \mathcal{B} = \{0\}$;
(c) $\mathcal{A} = \{0, 1, 2, 3, \ldots\}$, $\mathcal{B} = \{0, 2, 4, 6, \ldots\}$, $\mathcal{A} \cap \mathcal{B} = \{0, 2, 4, 6, \ldots\}$;
The intersection cannot be uncountable as then we would have a subset of set $\mathcal{A}$ having a greater cardinality than the set $\mathcal{A}$ itself.

(iii) the set difference of two countably infinite sets $\mathcal{A}$ and $\mathcal{B}$ must be a countably set, which is to say it could be finite or countably infinite. For example:
(a) $\mathcal{A} = \mathbb{N}$, $\mathcal{B} = \mathbb{N}$, $\mathcal{A} - \mathcal{B} = \varnothing$;
(b) $\mathcal{A} = \{0, 1, 2, 3, \ldots\}$, $\mathcal{B} = \{1, 2, 3, \ldots\}$, $\mathcal{A} - \mathcal{B} = \{0\}$;
(c) $\mathcal{A} = \{0, 1, 2, 3, \ldots\}$, $\mathcal{B} = \{0, 2, 4, 6, \ldots\}$, $\mathcal{A} - \mathcal{B} = \{1, 3, 5, 7, \ldots\}$;
The set difference cannot be uncountable as then we would have a subset of set $\mathcal{A}$ having a greater cardinality than the set $\mathcal{A}$ itself.