Université d'Ottawa
Faculté de génie

École de science informatique
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L'Université canadienne
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University of Ottawa
Faculty of Engineering

School of Electrical Engineering
and Computer Science

ELG 3126 **RANDOM SIGNALS AND SYSTEMS** **Winter 2023**

ASSIGNMENT 3

(due at 8:30 AM Thursday, Feb. 2 in class)

1. Eight rooks are placed on a chess board at random. Twelve squares on the chessboard are marked. What is the probability that they are all in marked places?

2. In a certain random experiment, the result can be any non-zero integer number, and the probability that the outcome is i is $(\frac{1}{3})^{|i|}$ for all nonzero integers i. What is the probability that one of -1, 2 or 5 is chosen? What is the probability that a multiple of 4 (positive or negative) is chosen?

3. A toddler pulls three volumes of an encyclopedia from a bookshelf and, after being scolded, places them back at random. What is the probability that the books are placed back in their correct order?

4. A multiple choice test has 10 questions, each of which has five possible answers from which to select the correct response. How many ways are there to answer the questions? If we assume that students pick their answers entirely at random, what is the probability that two particular students have the same set of answers other than on three (unspecified) questions. Assume that the two papers are both answered independently, i.e., no copying answers takes place. (Hopefully you would not answer questions at random on any test in your courses, and thus this is an unrealistic assumption for real tests—but questions would be answered independently.)

5. Ten different people are separately asked to pick a number from 1 to 10 at random. Assuming the people all comply, what is the probability that each number is picked once, i.e., that each person selects a different number. If only nine people pick numbers, what is the chance that they all pick different numbers?

6. Assume that if we pick a person at random from a very large group and determine the day of the week on which they were born, there is an equal chance that it is each possibility (Monday, Tuesday, etc.). Suppose seven people are chosen from the group at random. What is the chance that no two persons were born on the same day of the week?

7. Vincent Massey Park is home to many raccoons and it is desired to learn just how many there are. It is exceedingly difficult to determine this value exactly, so it is decided to resort to a probabilistic approach. For this, 12 raccoon are caught in the park, tagged and then released. A few days later, 30 raccoons are caught (you may assume that these are 30 raccoons chosen at random from the population of raccoons in the park), and it is found that six of the raccoons are from the previously tagged group. Find the probability that if there are N raccoons, that six of the 30 caught would be tagged raccoons. If this value is denoted p_N, one way to estimate the population is to find that value for N that causes p_N to be the largest (i.e., the value of N that makes it most likely that six of the 30 caught raccoons would be tagged); this is known as the *maximum likelihood estimate of the parameter N*. What is that value?
   Hint: Consider the ratio p_N/p_{N-1} compared to 1.

. . . *(over)*

8. A circuit board requires the use of four different ICs that all in the same package form. Unfortunately, when four ICs are given to be placed on the board, the IC identification indicating which IC is which is undecipherable. The circuit board assembler thus arbitrarily places the ICs in the four sockets (A, B, C, and D) hoping for the best. What is the probability that (i) at least socket A gets the correct IC; (ii) at least socket A and socket B get the proper IC; (iii) at least one socket gets the correct IC; and (iv) no IC is placed in the correct socket?

9. In class we showed that if $\mathcal{P}(\mathcal{A}) = 0$, then $\mathcal{A}$ and any other event $\mathcal{B}$ are independent just as the null event $\varnothing$ and any event $\mathcal{B}$ are independent. We also note that $\mathcal{S}$ and any $\mathcal{B}$ are independent, so we might conjecture that *an event $\mathcal{A}$ for which $\mathcal{P}(\mathcal{A}) = 1$ is independent of any event $\mathcal{B}$.* Prove this statement is true, or give an example where it is not.

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SOLUTIONS

1/ There are a total of 64!/56! ways in which we can pick 8 locations on a chess board, order being noted. There are 12!/4! ways in which we can pick from the 12 marked locations. Thus the probability of choosing only from 12 marked squares is

$$\frac{12! \times 56!}{4! \times 64!} = \frac{45}{491796152} \simeq 1.1183 \times 10^{-7}.$$

2/ $\mathcal{P}(\text{“}-1\text{” or “}2\text{” or “}5\text{”}) = \mathcal{P}(\text{“}-1\text{”}) + \mathcal{P}(\text{“}2\text{”}) + \mathcal{P}(\text{“}5\text{”}) = \frac{1}{3} + \frac{1}{9} + \frac{1}{243} = \frac{109}{243} \simeq 0.448560.$

$$\mathcal{P}(4 \text{ multiple}) = \mathcal{P}\Big(\bigcup_{\substack{i=-\infty \\ i\neq 0}}^{\infty} \{\text{“}4i\text{”}\}\Big) = 2\sum_{i=1}^{\infty} \mathcal{P}(\{\text{“}4i\text{”}\}) = 2\sum_{i=1}^{\infty} \frac{1}{3^{4i}} = 2\sum_{i=1}^{\infty} \frac{1}{81^i} = \frac{\frac{2}{81}}{1-\frac{1}{81}} = \frac{1}{40}.$$

3/ There are $3! = 6$ ways the books may be placed on the shelf and only one of these ways is the original ordering. Thus the chance the books are out back where they belong is $\frac{1}{6}$.

4/ There are 5^{10} possible ways to answer the questions on the test. Imagine we pick one particular sequence of possible answers. The number of ways to pick answers that match the particular selection on all questions except three particular identified questions is just $4^3 = 64$. There are 10!/(7!3!)=120 possibilities for selecting three of the 10 questions for the mismatch, so there are thus $120 \times 64 = 7680$ ways to answer where there are exactly three mismatched questions. If we consider the sequence of possible answers is the way the first student answers the questions, then there are just 7680 out of 5^{10} ways for the second student to answer and have exactly three mismatches. The chance one student answers the same as another identified student except for exactly three questions is thus $7680/5^{10} = \frac{1536}{1953125} = 7.86432 \times 10^{-4}$.

5/ There are 10^{10} possible ordered lists of the numbers chosen, and of these, 10! involve different numbers being selected (the number of ways of ordering the numbers 1 through 10). Thus the probability we seek is

$$\frac{10!}{10^{10}} = \frac{9!}{10^9} = \frac{567}{1562500} \simeq 3.628 \times 10^{-4}.$$

If only 9 people pick numbers, then there are 10^9 possible ordered lists of the numbers chosen. There are 9! ways of picking a specific set of 9 different number, and there are 10 possible sets of 9 different digits and so there are still $10 \times 9! = 10!$ ways of picking 9 different numbers for the event so the probability of all nine choices being different is

$$\frac{10!}{10^9} = \frac{9!}{10^8} = \frac{567}{156250} \simeq 3.628 \times 10^{-3}.$$

6/ If we select seven people in order and each can be born on one of seven days of the week, there are 7^7 possible lists of days of the week that could result. There are 7! ways that we could list the seven days without any repetition. As all singletons have the same probability here (the stated assumption we can make), the probability that no two selected people have birthdates on the same day of the week is $7!/7^7 = 6!/7^6 = 720/117649 \simeq 6.120 \times 10^{-3}$.

7/ Let N be the actual number of raccoons in the park. There are 12 tagged raccoons, and $N-12$ raccoons without tags. There are $\binom{N}{30} = \frac{N!}{30!(N-30)!}$ possible groups of 30 raccoons that could be selected (assumes $N \geq 30$). If the second captured group is to contain k tagged raccoons, (for $k \in \{0, 1, 2, \ldots, 10\}$), it means we must capture k from the tagged raccoon group and $30-k$ from the $N-12$ untagged raccoons. There are thus $\binom{12}{k}\binom{N-12}{30-k}$ possible groups of 30 raccoons with k of them tagged. Thus the probability of having k tagged raccoons in the sample is $\binom{12}{k}\binom{N-12}{30-k}/\binom{N}{30}$ which for $k=6$ is

$$p_N = \frac{12!\,(N-12)!\,(N-30)!\,30!}{6!\,6!\,24!\,(N-36)!\,N!} = \frac{12!\,30!}{6!\,6!\,24!}\frac{(N-12)!\,(N-30)!}{N!\,(N-36)!}.$$

We need to find the value of N that maximizes this, and to do that we may note that

$$\frac{p_N}{p_{N-1}} = \frac{(N-12)(N-30)}{N(N-36)} = \frac{N^2-42N+360}{N^2-36N}.$$

This ratio is greater than 1 when $N^2-42N+360 > N^2-36N$ or more simply when $360 > 6N$, implying that p_N is an increase function of N when $N < 60$, and decreasing when $N > 60$. Thus as N is increased from 25, p_N increases until it reached a peak at $N = 59$, stays the same for $N = 60$, and then decreases for larger N. The maximum likelihood estimate of N is thus 59 or 60 (either value is as likely as the other so either one can be used).

8/ There are 4! ways to fill 4 sockets with 4 distinct ICs. There are 3! ways that socket A can be correctly filled, and 2! that both sockets A and B can be correctly filled. Thus

(i) $\mathcal{P}(\text{socket A correctly filled}) = 3!/4! = \frac{1}{4}$ (ii) $\mathcal{P}(\text{sockets A an B correctly filled}) = 2!/4! = \frac{1}{12}$.

To answer (iii), we may proceed as follows: The number of ways that k identified sockets can be filled is clearly $(4-k)!$. Let $\mathcal{A}_1$ be the event where the ith socket gets the correct IC. Clearly $\mathcal{P}(\mathcal{A}_i) = \frac{1}{4}$ (same answer as in (i)) for all i, and $\mathcal{P}(\mathcal{A}_i \cap \mathcal{A}_j) = \frac{1}{12}$ (same answer as in (ii)) for all $i, j, i \neq j$ [there are $\binom{4}{2} = 6$ such cases]. Likewise $\mathcal{P}(\mathcal{A}_i \cap \mathcal{A}_j \cap \mathcal{A}_k) = 1!/4! = \frac{1}{24}$ for all i, j, k when they are all different [there are $\binom{4}{3} = 4$ such cases], and $\mathcal{P}(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4) = \frac{1}{24}$. The probability that at least one IC is in it correct socket is

$$\begin{aligned}\mathcal{P}(\bigcup_{i=1}^{4}\mathcal{A}_i) &= \sum_{i=1}^{4}\mathcal{P}(\mathcal{A}_i) - \sum_{\substack{i,j\\ i<j}}\mathcal{P}(\mathcal{A}_i \cap \mathcal{A}_j) \sum_{\substack{i,j,k\\ i<j<k}}\mathcal{P}(\mathcal{A}_i \cap \mathcal{A}_j \cap \mathcal{A}_k) + -\mathcal{P}(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4)\\ &= 4 \times \tfrac{1}{4} - 6 \times \tfrac{1}{12} + 4 \times \tfrac{1}{24} - \tfrac{1}{24} = \tfrac{5}{8} = 0.625 \quad \text{[the answer to (iii)]}.\end{aligned}$$

Thus the probability that no socket gets the correct IC is $1 - \mathcal{P}(\text{at least one socket has corr. IC}) = \frac{3}{8} = 0.375$,which is then the answer to (iv).

9/ The conjecture is true: if $\mathcal{A}$ is an almost certain event and $\mathcal{B}$ any other event, then they are necessarily independent. *Proof:* For any events $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{B} = (\mathcal{A} \cap \mathcal{B}) \cup (\overline{\mathcal{A}} \cap \mathcal{B})$. Since $\mathcal{A} \cap \mathcal{B}$ and $\overline{\mathcal{A}} \cap \mathcal{B}$ are disjoint events, we see that

$$\mathcal{P}(\mathcal{B}) = \mathcal{P}(\mathcal{A} \cap \mathcal{B}) + \mathcal{P}(\overline{\mathcal{A}} \cap \mathcal{B}).$$

Now if $\mathcal{P}(\mathcal{A}) = 1$, then $\mathcal{P}(\overline{\mathcal{A}}) = 0$, and since $\overline{\mathcal{A}} \cap \mathcal{B} \subset \overline{\mathcal{A}}$, it follows that $\mathcal{P}(\overline{\mathcal{A}} \cap \mathcal{B}) = 0$, giving us that

$$\mathcal{P}(\mathcal{B}) = \mathcal{P}(\mathcal{A} \cap \mathcal{B}).$$

But since $\mathcal{P}(\mathcal{A}) = 1$, this can be written as

$$\mathcal{P}(\mathcal{A})\mathcal{P}(\mathcal{B}) = \mathcal{P}(\mathcal{A} \cap \mathcal{B}).$$

establishing that $\mathcal{A}$ and $\mathcal{B}$ are independent whenever $\mathcal{P}(\mathcal{A}) = 1$.