Université d'Ottawa
Faculté de génie

École de science informatique
et de génie électrique

uOttawa

L'Université canadienne
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University of Ottawa
Faculty of Engineering

School of Electrical Engineering
and Computer Science

ELG 3126 **RANDOM SIGNALS AND SYSTEMS** **Winter 2023**

ASSIGNMENT 4

(due at 8:30 AM Thursday, Feb. 9 in class)

1. A certain data source produces messages that can be one of five possibilities selected from the set $\{A, B, C, D, E\}$. The random experiment consists of observing the value that the data source actually produces. The probability of the five possibilities being generated is

$$\mathcal{P}(\{A\}) = \tfrac{1}{2}, \quad \mathcal{P}(\{B\}) = \tfrac{1}{4}, \quad \mathcal{P}(\{C\}) = \tfrac{1}{8} \quad \text{and } \mathcal{P}(\{D\}) = \mathcal{P}(\{E\}) = \tfrac{1}{16}.$$

The message produced is to be transmitted over a binary communication system, and to do so it is necessary to represent every message using codeword composed of a string of binary values. The code chosen is

$$A \rightarrow 1 \qquad B \rightarrow 01 \qquad C \rightarrow 001 \qquad D \rightarrow 0001 \qquad E \rightarrow 0000.$$

A random variable $\mathbf{Y}$ is defined that corresponds to the length of the binary codeword produced in this situation.

(a) The sample space for this experiment is $\mathcal{S} = \{A, B, C, D, E\}$. What is the mapping on $\mathcal{S}$ that is the random variable $\mathbf{Y}$?

(b) What is the event "$\mathbf{Y} \leq 2$", and what is the probability of this event?

2. A certain random experiment consists of picking a point (x, y) on the Cartesian plane from the square $[0, 1] \times [0, 1]$. The selection is made "entirely at random." A random variable $\mathbf{Z}$ is defined for this experiment to be the mapping given by $\mathbf{Z}((x, y)) = x^2 + y^2$.

(a) What possible values can $\mathbf{Z}$ be?

(b) Sketch the region on the plane corresponding to the event $\mathbf{Z} \leq z$ (for all $z \in \mathbb{R}$), and what is the probability of this event.

(c) Suppose $\mathcal{A} = \{(x, y) | \frac{1}{4} < x \leq \frac{1}{3}, \frac{1}{5} < y < \frac{3}{5}\}$. What value must the probability that the outcome is in $\mathcal{A}$ be? Some intuitive reasoning may be used.

3. From the text: 3.19, but make $p = 0.005$ and assume that each bit is repeated just three times.

4. An information source produces a pair of binary symbols with four possible values: 00, 01, 10,and 11. A random variable $\mathbf{X}$ is defined by mapping one binary pair value to 1, another to 2, a third to 3, and the last to 4. Let $p_k \triangleq \mathcal{P}(\mathbf{X} = k)$ for $k = 1, 2, 3, 4$.

(a) Sketch the cdf of $\mathbf{X}$, $F_{\mathbf{X}}(x)$, for each of the three cases

(i) $p_k = p_1/(k + 1)$ for $k = 2, 3, 4$,

(ii) $p_{k+1} = p_k/3$ for $k = 1, 2, 3$,

(iii) $p_{k+1} = p_k/2^k$ for $k = 1, 2, 3$.

(b) Use the cdf to find $\mathcal{P}(\mathbf{X} \leq -1)$, $\mathcal{P}(\mathbf{X} < \frac{3}{2})$, $\mathcal{P}(0.5 < \mathbf{X} \leq 2.999)$, $\mathcal{P}(1 < \mathbf{X} \leq 4)$.

. . . *(over)*

5. A dart is equally likely to land at any point inside a circular target of radius 3 units. A random variable $\mathbf{R}$ is define to be the distance the point where the dart lands is away from the target centre.
   (a) What is the sample space for this experiment and what is the mapping from the sample space to $\mathbb{R}$ that $\mathbf{R}$ represents
   (b) The "bull's eye" is a central circular area on the target with radius 0.25 units. Let $\mathcal{A}$ be the event corresponding to "dart hits the bull's eye." Express this event as $\{\mathbf{R} \in \mathcal{I}\}$ for a suitable set $\mathcal{I}$, and find the probability of this event.
   (c) Find and sketch the probability distribution function (the cumulative distribution function) of $\mathbf{R}$.
6. From the text: 4.14, 4.24
7. The random variable $\mathbf{X}$ has the cdf

$$F_{\mathbf{X}}(x) = \begin{cases} 0, & x < 0; \\ 1 - \cos^2(\pi x/2), & 0 \leq x < 1; \\ 1, & x \geq 1. \end{cases}$$

   (a) What is $\mathcal{P}(\mathbf{X} > \frac{1}{3})$?
   (b) What is the probability density function of $\mathbf{X}$?
   (c) Classify $\mathbf{X}$ as being discrete, continuous or of mixed type.

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ASSIGNMENT 4

SOLUTIONS

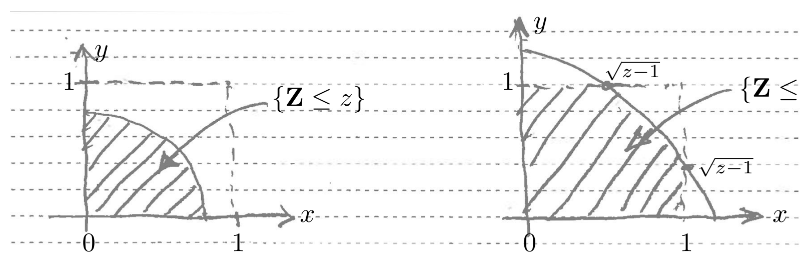
1/ (a) The random variable is defined on $\mathcal{S}$ as $\mathbf{Y}(A) = 1$, $\mathbf{Y}(B) = 2$, $\mathbf{Y}(C) = 3$, $\mathbf{Y}(D) = 4$, $\mathbf{Y}(E) = 4$, Obviously $\mathbf{Y}$ can have value 1, 2, 3 or 4, and nothing else.

(b) The event $\{\mathbf{Y} \le 2\}$ is $\{A, B\}$, which has probability

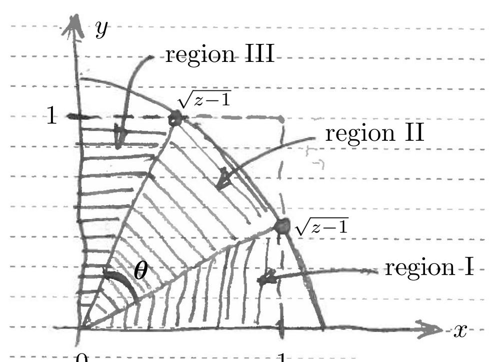
$$\mathcal{P}(\{A, B\}) = \mathcal{P}(\{A\}) + \mathcal{P}(\{B\}) = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}.$$

2/ (a) Since $x, y \in [0, 1]$, the value of $x^2 + y^2$ can be any value from 0 to 2, i.e., the set of possible values for $\mathbf{Z}$ is $[0, 2]$.

(b) The region on the Cartesian plane where $x^2 + y^2 \le z$ is the set of points inside the disc of radius $\sqrt{z}$ when $z \ge 0$ and is the empty set if $z < 0$. Thus the event $\{\mathbf{Z} \le z\}$ is the overlap of this region and the $[0, 1] \times [0, 1]$ square. This is the null event is $z < 0$ is the entire $[0, 1] \times [0, 1]$ square if $z \ge 2$ and otherise is otherwise one of the two regions shown below:



The probabilities of these events are just the areas of the regions which is 0 for $z \le 0$, is 1 for $z \ge 2$, while for $z \in (0, 1]$ is $\frac{1}{4}\pi z$. For $z \in [1, 2)$ we have to work a bit harder to find the area:



The area of region I and of region III each $\frac{1}{2}\sqrt{z-1}$, while region II is a sector of a disc of radius $\sqrt{z}$ subtended by the angle $\theta = \frac{1}{2}\pi - \tan^{-1}(\sqrt{z-1}) = \tan^{-1}(1/\sqrt{z-1})$ has area $\pi z \frac{\theta}{2\pi} = \frac{1}{2} z \tan^{-1}(1/\sqrt{z-1})$. The total are is then $\sqrt{z-1} + \frac{1}{2} z \tan^{-1}(1/\sqrt{z-1})$. This then gives us the overall result:

$$F_{\mathbf{Z}}(z) = \begin{cases} 1, & z \ge 2; \\ \sqrt{z-1} + \frac{1}{2} z \tan^{-1}(1/\sqrt{z-1}) & 1 < z < 2; \\ \frac{1}{4}\pi z & 0 < z \le 1; \\ 0, & z \le 0. \end{cases}$$

. . . *(problem continued on the next page)*

2/ (continued)

(c) The area of the set $\mathcal{A}$ is $\frac{2}{5} \times \frac{1}{12} = \frac{1}{30}$ and $\mathcal{A}$ is entirely within $\mathcal{S}$ so $\mathcal{P}(\mathcal{A}) = \frac{1}{30}$.

3/ (3.19-modified) If a majority vote is taken, then an error after decoding occurs when there are two or three channel errors during transmission of the three bits, Thus

$$\mathcal{P}(\text{decoding error}) = \sum_{k=2}^{3} \mathcal{P}(k \text{ transmission errors}) = \sum_{k=2}^{3} \binom{3}{k} p^k (1-p)^{3-k}$$
$$= \binom{3}{2}(.005)^2(.995)^1 + \binom{3}{3}(.005)^3(.995)^0 = 7.475 \times 10^{-5}.$$

4/ (a) The values of the probabilities arise from $p_1 + p_2 + p_3 + p_4 = 1$.

For (i), this gives $p_1 + \frac{1}{3}p_1 + \frac{1}{4}p_1 + \frac{1}{5}p_1 = 1$ or $\frac{107}{60}p_1 = 1$, giving us $p_1 = \frac{60}{107}$, $p_2 = \frac{20}{107}$, $p_3 = \frac{15}{107}$, $p_4 = \frac{12}{107}$.

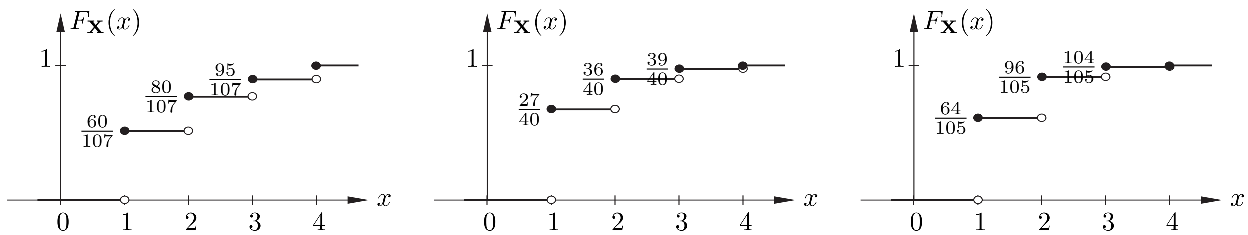
For (ii), we get $p_1 + \frac{1}{3}p_1 + \frac{1}{9}p_1 + \frac{1}{27}p_1 = 1$ or $\frac{40}{2}7p_1 = 1$, giving us $p_1 = \frac{27}{40}$, $p_2 = \frac{9}{40}$, $p_3 = \frac{3}{40}$, $p_4 = \frac{1}{40}$.

For (iii), we get $p_1 + \frac{1}{2}p_1 + \frac{1}{8}p_1 + \frac{1}{64}p_1 = 1$ or $\frac{105}{64}p_1 = 1$, giving us $p_1 = \frac{64}{105}$, $p_2 = \frac{32}{105}$, $p_3 = \frac{8}{105}$, $p_4 = \frac{1}{105}$.

The cdf in all these cases is

$$F_{\mathbf{X}}(x) = \begin{cases} 0, & x < 1; \\ p_1, & 1 \le x < 2; \\ p_1 + p_2, & 2 \le x < 3; \\ p_1 + p_2 + p_3, & 3 \le x < 4; \\ 1, & x \ge 4; \end{cases}$$

The sketches are thus



case (i) case (ii) case (iii)

(b) $\mathcal{P}(\mathbf{X} \le -1) = F_{\mathbf{X}}(-1) = 0$ in all these cases.

$\mathcal{P}(\mathbf{X} < 1.5) = F_{\mathbf{X}}(1.5 - 0) = p_1$ in all these, which has value $\frac{60}{107}$ for (i), value $\frac{27}{40}$ for (ii), and value $\frac{64}{105}$ for (iii).

$\mathcal{P}(0.5 < \mathbf{X} \le 2.999) = F_{\mathbf{X}}(2.99) - F_{\mathbf{X}}(0.5) = p_1 + p_2$ in all these, which has value $\frac{80}{107}$ for (i), value $\frac{9}{10}$ for (ii), and value $\frac{32}{35}$ for (iii).

$\mathcal{P}(1 < \mathbf{X} \le 4) = F_{\mathbf{X}}(4) - F_{\mathbf{X}}(1) = 1 - p_1$ in all theses, which has value $\frac{47}{107}$ for (i), value $\frac{13}{40}$ for (ii), and value $\frac{41}{105}$ for (iii).

5/ (a) The outcomes may be described as the set of (x, y) coordinates for the place the dart lands, and so the sample space can be all points inside a disc of radius 3 centered at the origin: $\mathcal{S} = \{(x, y) \mid x^2 + y^2 \le 9\}$. $\mathbf{R}((x, y)) = \sqrt{x^2 + y^2}$.

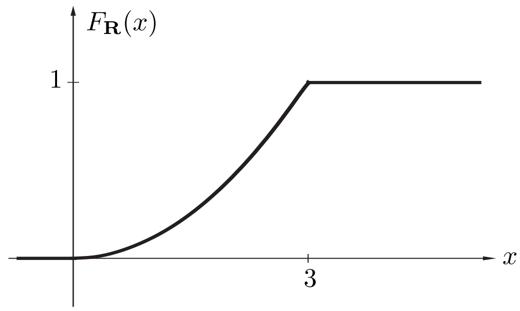
(b) The bull's eye corresponds to the event $\mathcal{A} = \{(x, y) \mid \sqrt{x^2 + y^2} \le \frac{1}{4}\}$ which is thus $\{\mathbf{R} \le \frac{1}{4}\}$, i.e., $\{\mathbf{R} \in [0, \frac{1}{4}]\}$ which makes the set $\mathcal{I}$ mentioned in the question to be $\mathcal{I} = [0, \frac{1}{4}]$, but this is not unique as we could also say for example $\mathcal{A} = \{\mathbf{R} \in (-\infty, \frac{1}{4}]\}$ (since $\mathbf{R}$ is always non-negative). The probability of the event is the fraction of the total area of the whole target that the bull's eye represents:

$$\mathcal{P}(\mathcal{A}) = \frac{\pi(\frac{1}{4})^2}{\pi(3)^2} = \frac{1}{144}.$$

. . . (problem continued on the next page)

5/ (continued)

(c)
$$F_{\mathbf{R}}(x) = \mathcal{P}(\mathbf{R} \le x) = \begin{cases} 1, & \text{if } x > 3; \\ \dfrac{\pi x^2}{\pi (3)^2} = \frac{1}{9}x^2, & \text{if } 0 \le x \le 3; \\ 0, & \text{if } x < 0. \end{cases}$$



6/ (4.14), (4.24)

[4.14](a) As $F_{\mathbf{X}}(x)$ is neither a step function nor a continuous function everywehere (it has a jump at $x = -1$ and $x = 0$), $\mathbf{X}$ is a random variable of mixed type.

(b) (i) $\mathcal{P}(\mathbf{X} < -1) = F_{\mathbf{X}}(-1 - 0) = 0$;

(ii) $\mathcal{P}(\mathbf{X} \le -1) = F_{\mathbf{X}}(-1) = \frac{1}{5}$;

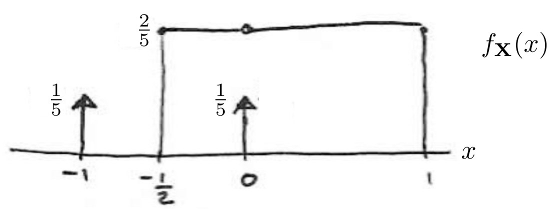
(iii) $\mathcal{P}(-1 < \mathbf{X} < -\frac{3}{4}) = \mathcal{P}(\mathbf{X} < -\frac{3}{4}) - \mathcal{P}(\mathbf{X} \le -1) = F_{\mathbf{X}}(-\frac{3}{4} - 0) - F_{\mathbf{X}}(-1) = \frac{1}{5} - \frac{1}{5} = 0$;

(iv) $\mathcal{P}(-\frac{1}{2} \le \mathbf{X} < 0) = \mathcal{P}(\mathbf{X} < 0) - \mathcal{P}(\mathbf{X} < -\frac{1}{2}) = F_{\mathbf{X}}(0 - 0) - F_{\mathbf{X}}(-\frac{1}{2} - 0) = \frac{2}{5} - \frac{1}{5} = \frac{1}{5}$;

(v) $\mathcal{P}(-\frac{1}{2} \le \mathbf{X} \le \frac{1}{2}) = \mathcal{P}(\mathbf{X} \le \frac{1}{2}) - \mathcal{P}(\mathbf{X} < -\frac{1}{2}) = F_{\mathbf{X}}(\frac{1}{2}) - F_{\mathbf{X}}(-\frac{1}{2} - 0) = \frac{4}{5} - \frac{1}{5} = \frac{3}{5}$;

(vi) $\mathcal{P}(|\mathbf{X} - \frac{1}{2}| < \frac{1}{2}) = \mathcal{P}(0 < \mathbf{X} < 1) = \mathcal{P}(\mathbf{X} < 1) - \mathcal{P}(\mathbf{X} \le 0) = F_{\mathbf{X}}(1 - 0) - F_{\mathbf{X}}(0) = 1 - \frac{3}{5} = \frac{2}{5}$.

[4.24](a) $f_{\mathbf{X}}(x) = \frac{1}{5}\delta(x+1) + \frac{1}{5}\delta(x) + \frac{2}{5}[u(x + \frac{1}{2}) - u(x - 1)]$.



(b) (i) $\mathcal{P}(-1 \le \mathbf{X} < \frac{1}{2}) = \displaystyle\int_{-1}^{1/2} f_{\mathbf{X}}(x)\,dx = \int_0^{1/2} \tfrac{1}{2}e^{-2x}\,dx = 1 - \tfrac{1}{4}e^{-1}$.

(i) $\mathcal{P}(\mathbf{X} < -1) = \int_{-\infty}^{-1-0} f_{\mathbf{X}}(x)\,dx = 0$ (the impulse at $x = -1$ is excluded;

(ii) $\mathcal{P}(\mathbf{X} \le -1) = \int_{-\infty}^{-1} f_{\mathbf{X}}(x)\,dx = \frac{1}{5}$ (the impulse at $x = -1$ is included ;

(iii) $\mathcal{P}(-1 < \mathbf{X} < -\frac{3}{4}) = \int_{-1+0}^{-\frac{3}{4}-0} f_{\mathbf{X}}(x)\,dx = \int_{-1+0}^{-\frac{3}{4}-0} 0\,dx = 0$;

(iv) $\mathcal{P}(-\frac{1}{2} \le \mathbf{X} < 0) = \int_{-\frac{1}{2}}^{0-0} f_{\mathbf{X}}(x)\,dx = \int_{-\frac{1}{2}}^{0-0} \frac{2}{5}\,dx = \frac{1}{5}$;

(v) $\mathcal{P}(-\frac{1}{2} \le \mathbf{X} \le \frac{1}{2}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_{\mathbf{X}}(x)\,dx = \frac{1}{5} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2}{5}\,dx = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$;

(vi) $\mathcal{P}(|\mathbf{X} - \frac{1}{2}| < \frac{1}{2}) = \mathcal{P}(0 < \mathbf{X} < 1) = \int_{0+0}^{1-0} f_{\mathbf{X}}(x)\,dx = \int_0^1 \frac{2}{5}\,dx = \frac{2}{5}$.

. . . *(continued on the next page)*

7/ (a) $\mathcal{P}(\mathbf{X} > \frac{1}{3}) = 1 - \mathcal{P}(\mathbf{X} \leq \frac{1}{3}) = 1 - F_{\mathbf{X}}(\frac{1}{3}) = \cos^2(\pi/6) = (\frac{1}{2}\sqrt{3})^2 = \frac{3}{4}$.

(b) $f_{\mathbf{X}}(x) = \dfrac{d}{dx}F_{\mathbf{X}}(x) = \begin{cases} 0, & \text{if } x \leq 0 \text{ or } x \geq 1; \\ \pi\cos(\frac{1}{2}\pi x)\sin(\frac{1}{2}\pi x) = \frac{1}{2}\pi\sin(\pi x), & \text{if } 0 < x < 1. \end{cases}$

There are no impulses as $F_{\mathbf{X}}(x)$ has no jump points (and no corner points). The derivative of $F_{\mathbf{X}}(x)$ exists everywhere in a normal sense.

(c) $F_{\mathbf{X}}(x)$ is a continuous function, hence $\mathbf{X}$ is a continuous random variable.