Université d'Ottawa
Faculté de génie

École de science informatique
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University of Ottawa
Faculty of Engineering

School of Electrical Engineering
and Computer Science

ELG 3126 **RANDOM SIGNALS AND SYSTEMS** **Winter 2023**

ASSIGNMENT 5

(due at 8:30 AM Thursday, Feb. 16 in class)

1. The random variable $\mathbf{X}$ has the cdf

$$F_{\mathbf{X}}(x) = \begin{cases} 0, & x < 0; \\ x^3, & 0 \le x < 1; \\ 1, & x \ge 1. \end{cases}$$

   (a) What is $\mathcal{P}(\mathbf{X} > \frac{2}{3})$?
   (b) What is the probability density function of $\mathbf{X}$?
   (c) Classify $\mathbf{X}$ as being discrete, continuous or of mixed type.

2. A random variable $\mathbf{X}$ has a probability density function

$$f_{\mathbf{X}}(x) = \begin{cases} c(5 - 2x + x^2), & \text{if } 0 < x < 2; \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the value of c and sketch $f_{\mathbf{X}}(x)$.
   (b) Find and sketch the cdf $F_{\mathbf{X}}(x)$.
   (c) Find $\mathcal{P}(\mathbf{X} = 1)$, $\mathcal{P}(0 < \mathbf{X} < \frac{1}{2})$, and $\mathcal{P}(|\mathbf{X} - \frac{2}{3}| < \frac{1}{3})$.

3. Let $\mathbf{X} \sim N(3, 4)$. Find numerically using the table of $Q(x)$ provided in class (a) $\mathcal{P}(\mathbf{X} > 4)$, (b) $\mathcal{P}(\mathbf{X} < 10)$, (c) $\mathcal{P}(4 < \mathbf{X} < 10)$, (d) $\mathcal{P}(\mathbf{X} = 2)$ and (e) $\mathcal{P}(|\mathbf{X} - 1| > 3)$.

4. One of the properties of the Gaussian distribution is that the "tails" of the distribution goes to zero very quickly as the argument of density function grows in magnitude, so much so it dominates any polynomial growth, which means the product of any polynomial (of finite order) and any Gaussian density function must tend to 0 as the argument grows in magnitude. To see this, prove the more basic result that for any $n \in \{0, 1, 2, 3, \ldots\}$,

$$\lim_{x \to \infty} x^n e^{-x^2/2} = 0.$$

   Hint: You will probably have to use L'Hospital's rule. It may help to first note that if $x > 2$,

$$0 < e^{-x^2/2} < e^{-x},$$

   and that it is easy to prove $\lim_{x \to \infty} x^n e^{-x} = 0$

5. In class we learned that the function $Q(x)$ for $x > 0$ was bounded by two simpler functions:

$$\left[1 - \frac{1}{x^2}\right] \frac{1}{x\sqrt{2\pi}} e^{-x^2/2} < Q(x) < \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}.$$

   Plot the Q-function accurately for x from 0 to 7. Use a log scale for the vertical axis. Plot on this same graph these upper and lower bounds. For what values of x would you say the upper bound

5. [*continued*]
is within 1% of $Q(x)$ [two significant figures of accuracy from the upper bound]. Determine $Q(10)$ numerically as accurately as you can from the approximation.

   It is expected that you will use the computer to make these plots. EXCEL can be used to evaluate $Q(x)$ with the built-in function NORMSDIST, as $Q(x) = \text{NORMSDIST}(-x)$. Previous versions of EXCEL would give $Q(10)$ (i.e., NORMSDIST(-10)) to be 0; is the version of EXCEL on your computer any better and does it now return an accurate (in terms of percentage error) answer? You will need to find a Table of values of $Q(x)$ from an authoritative source such as *The Handbook of Mathematical Functions* by Abramowitz and Stegun (available online) or from a table in the Wikipedia article on the "standard distribution" that gives $Q(10)$.

6. Prove that the area under the density function of a Gaussian random variable with $\sigma > 0$ is indeed one.
Hint: First show by a substition of variable that the area under the general Gaussian distribution is the same as the are under the density function of the standard form. Then if

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx,$$

   then

$$I^2 = \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy,$$

   which can be evaluated easily by a change to polar coordinates.

7. For the Rayleigh density function (as defined in class), find the mode of the distribution and the value of the density function at the mode. Find also the mean and median of the distribution.
Remark: The *mode* of a distribution is defined to be the value of the distribution with the highest likelihood. For a discrete random variable $\mathbf{X}$, this is the value where $\mathcal{P}(\mathbf{X} = x)$ is largest. For a continuous random variable, it is where the density function has its peak. (A density function with one local maximum is termed *uni-modal*, one with two local maxima is termed *bi-modal*, etc.) The *median* of a distribution is the smallest value where the distribution function is greater than or equal to 0.5. For a continuous random variable, the median is then the value where the probability is one-half of being below it, and one-half above it (the so-called 50th percentile). The mean, median and mode are three measures of where the "centre" of the distribution lies.

8. A Christmas fruitcake produced at a large bakery has numbers of fruity bits that randomly vary from cake to cake. Specifically, it has Poisson-distributed independent numbers of sultana raisins, iridescent red cherry bits, and radioactive green cherry bits with respective means of 48, 24 and 12 bits per cake. Suppose you are offered and politely accept a slice that is 1/12 of the cake.
   (a) What is the probability that you are fortunate enough to get no green bits in your slice?
   (b) What is the probability that you are lucky and have no green bits and less than three red bits?
   (c) What is the probability that you are extremely lucky and have no green bits or red bots and have more than five raisins in your slice?

   Hint: You may assume that if the numbers of elements added to the cake had a Poisson distribution, then each slice of the cake also contains a number of elements that follows a Poisson distribution, whether the elements are raisins, green bits or red bits.
Remark: You can imagine a very large batch of mixture is made and then it is divided into cakes, so that how many of the fruity bits in each cake becomes random; each raisin has an equal chance of ending up in any one of the cakes with equal chance (that would be exactly so if raising had no volume, which is obviously not true), so the number of raisins in a given cake has a $B(n, p)$ distribution for a large number n, and so the Poisson distribution is a very good approximation. The same reasoning applies to green and red cherry bits.

9. Compare the Poisson approximation and binomial distribution probabilities of having value $k = 0, 1, 2, 3$ for the parameter choices $n = 10, p = 0.1$; $n = 20, p = 0.05$; and $n = 100, p = 0.01$.

10. An LCD display has a 1200×1000 pixels display. In the manufacturing process, defect can occur which render some pixels defective, but the display is still considered functional (saleable) if the display contains just a few defective pixels as a few defective pixels will hardly be noticeable. Suppose the probability that a pixel is defective is 10^{-5}, and the events that correspond to a specific pixel being defective are all independent. If the limit on defective pixels is set as 3, what fraction of manufactured displays will be considered acceptable? A numerical value is required.

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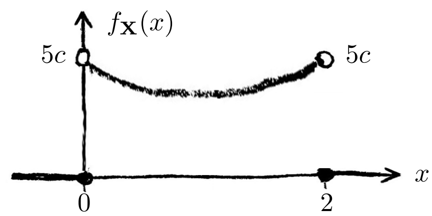
SOLUTIONS

1/ (a) $\mathcal{P}(\mathbf{X} > \frac{2}{3}) = 1 - \mathcal{P}(\mathbf{X} \le \frac{2}{3}) = 1 - F_{\mathbf{X}}(\frac{2}{3}) = 1 - (\frac{2}{3})^3 = \frac{19}{27} \simeq 0.7037.$

(b) $f_{\mathbf{X}}(x) = \begin{cases} \frac{d}{dx}F_{\mathbf{X}}(x), & \text{where } \frac{d}{dx}F_{\mathbf{X}}(x) \text{ exists;} \\ 0, & \text{otherwise;} \end{cases} = \begin{cases} 3x^2, & \text{if } 0 < x < 1. \\ 0, & \text{if } x \le 0 \text{ or } x \ge 1; \end{cases}$

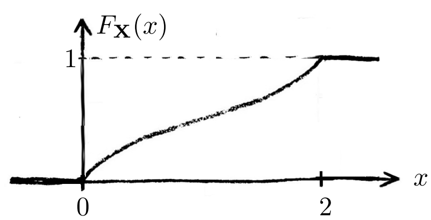
(c) $F_{\mathbf{X}}(x)$ is a continuous function, hence $\mathbf{X}$ is a continuous random variable.

2/ (a) $1 = \int_{-\infty}^{\infty} f(x)\,dx = c\int_0^2 5 - 2x + x^2\,dx = c\int_0^2 (x-1)^2 + 4\,dx = c[\frac{2}{3} + 8] = \frac{26}{3}c \implies c = \frac{3}{26} \simeq 0.115385.$



(b) $F_{\mathbf{X}}(x) = \int_{-\infty}^{x} f(\lambda)\,d\lambda = \begin{cases} 1, & x \ge 2; \\ 0, & x \le 0; \\ c\int_0^x 4 + (\lambda-1)^2\,d\lambda = c[4x + \frac{1}{3}[(x-1)^3 + 1], & 0 < x < 2; \end{cases}$

$$= \begin{cases} 1, & x \ge 2; \\ \frac{1}{26}x^3 - \frac{3}{26}x^2 + \frac{15}{26}x, & 0 < x < 2; \\ 0, & x \le 0. \end{cases}$$



(c) Since $\mathbf{X}$ is a continuous random variable, clearly $\mathcal{P}(\mathbf{X} = 1) = 0$.

$\mathcal{P}(0 < \mathbf{X} < \frac{1}{2}) = F_{\mathbf{X}}(\frac{1}{2}) - F_{\mathbf{X}}(0) = \frac{55}{208} \simeq 0.264423.$

$\mathcal{P}(|\mathbf{X} - \frac{2}{3}| < \frac{1}{3}) = \mathcal{P}(\frac{1}{3} < \mathbf{X} < 1) = F_{\mathbf{X}}(1) - F_{\mathbf{X}}(\frac{1}{3}) = \frac{1}{2} - \frac{1}{26}[(\frac{1}{3})^3 - 3(\frac{1}{3})^2 + 5] = \frac{112}{351} \simeq 0.319088.$

3/ Let $F(x)$ denote the cdf of a standard $N(0,1)$ random variable.

(a) $\mathcal{P}(\mathbf{X} > 4) = 1 - F(\frac{4-3}{2}) = Q(\frac{1}{2}) \simeq 0.3085.$

(b) $\mathcal{P}(\mathbf{X} < 10) = F(\frac{10-3}{2}) = 1 - Q(\frac{10-3}{2}) = 1 - Q(3.5) \simeq 0.9997674.$

(c) $\mathcal{P}(4 < \mathbf{X} < 10) = \mathcal{P}(\mathbf{X} < 10) - \mathcal{P}(\mathbf{X} < 4) < F(3.5) - F(\frac{1}{2}) = Q(.5) - Q(3.5) \simeq 0..3083$ (no more precision than this is possble using the table).

(d) $\mathcal{P}(\mathbf{X} = 2) = 0$ (since $\mathbf{X}$ is a continuous random variable).

(e) $\mathcal{P}(|\mathbf{X}-1| > 3) = \mathcal{P}(\mathbf{X} > 4) + \mathcal{P}(\mathbf{X} < -2) = Q(\frac{4-3}{2}) + 1 - Q(\frac{-2-3}{2}) = 1 + Q(\frac{1}{2}) - Q(-\frac{5}{2}) = Q(\frac{1}{2}) + Q(\frac{5}{2}) \simeq 0.3147$ (no more precision than this is possble using the table).

4/ For all $x \in \mathbb{R}$, $e^{-x} > 0$ and e^{-x} is monotonic decreasing. Since for $x > 2$, $\frac{1}{2}x^2 > x$, for $x > 2$ we see that

$$0 < x^n e^{-x^2/2} < x^n e^{-x} \qquad \text{for } x > 2.$$

Thus

$$0 \le \lim_{x\to\infty} x^n e^{-x^2/2} \le \lim_{x\to\infty} x^n e^{-x}.$$

But $x^n e^{-x} = \frac{x^n}{e^x}$, so by repeated application of l'Hospital's rule (n times), we have

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{(x^n)'}{(e^x)'} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

giving us that for any $n \in \mathbb{N}$,

$$0 \le \lim_{x\to\infty} x^n e^{-x^2/2} \le 0 \qquad \Longrightarrow \qquad \lim_{x\to\infty} x^n e^{-x^2/2} = 0.$$

5/ See the next page

6/ The area under the density function of a Gaussian random variable with parameters μ and σ^2 when σ^neq0 is

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\lambda-\mu)^2/2\sigma^2}\, d\lambda.$$

Using a substitution of variable $x = (\lambda - \mu)/\sigma$ so that $d\lambda$ is replaced by σdx, we get

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx.$$

Then

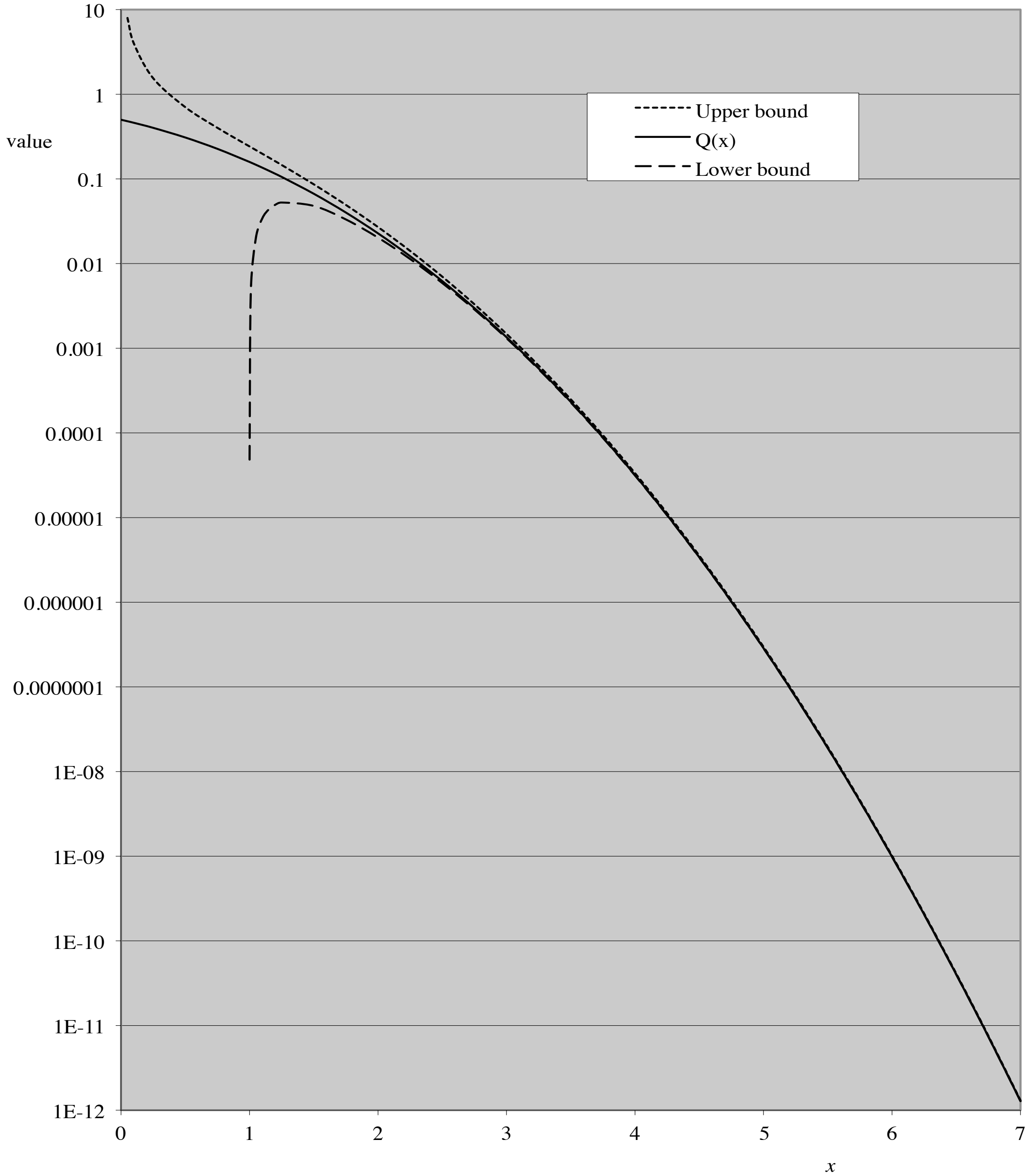
$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \int_{-\infty}^{\infty} e^{-y^2/2}\, dy = .\frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy.$$

Now this is an integral over the entire Cartesian plane, and if we transform this to polar coordinates using the subsitution $x = r\cos\theta$ and $y = r\sin\theta$ (so that $x^2 + y^2 = r^2$), then with $dx\,dy$ is replaced by $r\,dr\,d\theta$ and the entire plane being where $r \in [0,\infty)$ and $\theta \in [-\pi,\pi)$, then we see

$$I^2 = \frac{1}{2\pi} \int_0^{\infty} [\underbrace{\int_{-\pi}^{\pi} re^{-r^2/2}\, d\theta}_{2\pi\ re^{-r^2/2}}]\, dr = \int_0^{\infty} \underbrace{re^{-r^2/2}}_{-\frac{d}{dr}e^{-r^2/2}}\, dr = -e^{-r^2/2}\Big|_0^{\infty} = 1.$$

Since we know $I > 0$, it follows $I = 1$.

5/



By direct numerical evaluation and comparison, the upper bound is within 1% of the correct value when $x > 9.902$, while the lower bound is within 1% when $x > 3.950$. For $x = 10$, the bounds show that $Q(10)$ is between $\frac{0.99}{10\sqrt{2\pi}}e^{-50} \simeq 7.61765 \times 10^{-24}$ and $\frac{1}{10\sqrt{2\pi}}e^{-50} \simeq 7.69460 \times 10^{-24}$. [If we use the upper bound derived in Q5 on this assignment, we improve the upper bound to $\frac{0.9903}{10\sqrt{2\pi}}e^{-50} \simeq 7.61996 \times 10^{-24}$.] From more extensive tables, e.g., Table 26.2 in *The Handbook of Mathematical Functions* by Abramowitz and Stegun, we find that $Q(10)$ is accurately given as $10^{-23.11805} \simeq 7.620 \times 10^{-24}$. This table (and indeed the entirety of this famous handbook) is available on the web; a pdf file of it is available at `https://personal.math.sfu.ca/~cbm/aands/`; this handook is a classical reference on all sorts of mathematical functions. The Wikipedia article on the "Standard normal table" contains a table that states $Q(10) \simeq 7.61985 \times 10^{-24}$ but doesn't state the source for this value. Alternatively, we can now more conveniently turn to some software which will evaluate the function, though we may have to check the software's accuracy. Using the latest Microsoft Excel, `NORMSDIST(-10)` gives the value of $Q(10)$ as $7.61985302416048 \times 10^{-24}$ (but some older versions of Excel would just give the value 0). MATLAB gives the value through a built-in function `qfunc(x)` as $7.619853024160.59 \times 10^{-24}$ (very slighly different than Excel, but insignificantly so; this does show that numerical software does have its accuracy limitations.).

7/ For $f_{\mathbf{X}}(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2} u(x)$, the mean of $\mathbf{X}$ is given by

$$\begin{aligned}\mathcal{E}\{\mathbf{X}\} &= \int_0^\infty x \cdot \frac{x}{\alpha^2} e^{-x^2/2\alpha^2}\, dx = -\int_0^\infty x \cdot \frac{d}{dx}[e^{-x^2/2\alpha^2}]\, dx \\ &= \underbrace{x\, e^{-x^2/2\alpha^2}\Big|_0^\infty}_{0} + \int_0^\infty e^{-x^2/2\alpha^2}\, dx \\ &= \sqrt{2\pi\alpha^2} \underbrace{\int_0^\infty \frac{1}{\sqrt{2\pi\alpha^2}} e^{-x^2/2\alpha^2}\, dx}_{\frac{1}{2}} = \underline{\underline{\tfrac{1}{2}\sqrt{2\pi}\,\alpha \simeq 1.2533\alpha.}}\end{aligned}$$

For the mode, we seek the value of x that maximizes $f_{\mathbf{X}}(x)$, which is where, for $x > 0$, $f'_{\mathbf{X}}(x) = 0$:

$$f'_{\mathbf{X}}(x) = \frac{1}{\alpha^2}[e^{-x^2/2\alpha^2} - \frac{x^2}{\alpha^2} e^{-x^2/2\alpha^2}] = \frac{1}{\alpha^2}[1 - \frac{x^2}{\alpha^2}] e^{-x^2/2\alpha^2}, \qquad \text{for } x > 0,$$

which is zero only if $1 - \frac{x^2}{\alpha^2} = 0 \implies \underline{\underline{x = \alpha.}}$

For the median, we seek the value for x where $F_{\mathbf{X}}(x) = \frac{1}{2}$:

$$\text{for } x > 0, \qquad 1 - e^{-x^2/2\alpha^2} = \tfrac{1}{2} \implies e^{-x^2/2\alpha^2} = \tfrac{1}{2} \implies x^2 = 2\alpha^2 \ln 2 \implies \underline{\underline{x = \alpha\sqrt{2\ln 2} \simeq 1.1774\alpha.}}$$

8/ Let $\mathbf{N}_S$ be the number of raisins that are in your slice, let $\mathbf{N}_I$ be the number of iridescent bits that are in your slice, and let $\mathbf{N}_R$ be the number of radioactive bits that are in your slice. Clearly from the problem statement, $\mathbf{N}_S \sim P(\frac{48}{12} = 4)$, $\mathbf{N}_I \sim P(\frac{24}{12} = 2)$ and $\mathbf{N}_R \sim P(\frac{12}{12} = 1)$.

(a) $\mathcal{P}(\mathbf{N}_R = 0) = e^{-\mu}\frac{\mu^0}{0!}$ where $\mu = 1$,so $\mathcal{P}(\mathbf{N}_R = 0) = e^{-1} \simeq 0.36788$.

(b) $\mathcal{P}(\mathbf{N}_R = 0 \text{ and } \mathbf{N}_I \leq 2) = \mathcal{P}(\mathbf{N}_R = 0)\mathcal{P}(\mathbf{N}_I \leq 2)$ (since we know that these are independent events) $e^{-1} \times e^{-2}(1 + 2 + 2^2/2) = e^{-1} \times 5e^{-2} = 5e^{-3} \simeq 0.24894$.

(c)
$$\begin{aligned}\mathcal{P}(\mathbf{N}_R = 0 \text{ and } \mathbf{N}_I = 0 \text{ and } \mathbf{N}_S > 5)) &= \mathcal{P}(\mathbf{N}_R = 0)\mathcal{P}(\mathbf{N}_I = 0)\mathcal{P}(\mathbf{N}_S > 5) \text{(since they must be independent events)} \\ &= \mathcal{P}(\mathbf{N}_R = 0)\mathcal{P}(\mathbf{N}_I = 0)[1 - \mathcal{P}(\mathbf{N}_S \leq 5)] \\ &= e^{-1} \times e^{-2} \times (1 - e^{-4}\sum_{k=0}^{5} 4^k/k!) \\ &= e^{-3} - e^{-7} \times (1 + 4 + \frac{4^2}{2} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}) \\ &= e^{-3} - e^{-7} \times \frac{643}{15} \simeq 0.010698.\end{aligned}$$

9/

n= 10	p= 0.1	n*p= 1		
	k= 0	k= 1	k= 2	k= 3
Binomial (exact)	0.34868	0.38742	0.19371	0.05740
Poisson (approx)	0.36788	0.36788	0.18394	0.06131

n= 20	p= 0.05	n*p= 1		
	k= 0	k= 1	k= 2	k= 3
Binomial (exact)	0.35849	0.37735	0.18868	0.05958
Poisson (approx)	0.36788	0.36788	0.18394	0.06131

n= 100	p= 0.01	n*p= 1		
	k= 0	k= 1	k= 2	k= 3
Binomial (exact)	0.36603	0.36973	0.18486	0.06100
Poisson (approx)	0.36788	0.36788	0.18394	0.06131

10/ There are $1200 \times 1000 = 1.2 \times 10^6$ pixels in a display screen. Assuming the events that a particular pixel is defective are all independent, the number of pixels that are defective in the screens follow a binomial distribution $B(n, p)$ distribution with $n = 1.2 \times 10^6$ and $p = 10^{-5}$, so $np = 12$. The probability that there are k defective pixels is then $\binom{1.2 \times 10^6}{k}(10^{-5})^k(1 - 10^{-5})^{1.2\times 10^6 - k} \simeq e^{-12}\frac{(12)^k}{k!}$ (using the Poisson approximation). Hence

$$\begin{aligned}\mathcal{P}(\text{acceptable display}) &= \sum_{k=0}^{3}\binom{n}{k}p^k(1-p)^{n-k} \qquad \text{for } n = 1.2 \times 10^6,\ p = 10^{-5} \\ &\simeq e^{-12}\sum_{k=0}^{3}\frac{(12)^k}{k!} = 373\ e^{-12} \simeq \underline{\underline{0.0022918.}}\end{aligned}$$

Only 0.22918% of the manufactured screens would be acceptable. That could hardly be considered acceptable in a real factory with these parameters.