



**ELG 3126**

**RANDOM SIGNALS AND SYSTEMS**

**Winter 2023**

**ASSIGNMENT 6**

(due at 8:30 AM Thursday, March 2 in class)

1. If  $\mathbf{X} \sim U(-2, 4)$ , find the probability density function of the random variable  $\mathbf{Y} = 2\mathbf{X} + 3$ .
2. If  $\mathbf{X} \sim U(-2, 4)$ , find the probability density function of the random variable given by

$$\mathbf{Y} = \begin{cases} 1/(2\mathbf{X} + 1)^2, & \text{if } \mathbf{X} \neq -\frac{1}{2}; \\ 0, & \text{if } \mathbf{X} = -\frac{1}{2}. \end{cases}$$

3. If  $\mathbf{X} \sim U(-1, 4)$ , find the probability density function of the random variable  $\mathbf{Y} = \mathbf{X}^2 u(\mathbf{X})$ , where  $u(x)$  is the unit step function.  
*Hint:* First sketch  $g(x) = x^2 u(x)$ .

4. Let  $\mathbf{X}$  be a Gaussian random variable with mean 3 and variance 4. Find the pdf of the random variable  $\mathbf{Y} = g(\mathbf{X})$  if  $g(x) = xu(x) + x^3 u(-x)$ .  
*Hint:* First sketch  $g(x)$ .

5. If  $\mathbf{X}$  is a random variable with probability density function

$$f_{\mathbf{X}}(x) = 2e^{-2x} u(x),$$

Find the probability density function of  $\mathbf{Y} = e^{\mathbf{X}}$ , and then *using*  $f_{\mathbf{Y}}(y)$ , find the mean of  $e^{\mathbf{X}}$ .

6. Suppose  $\mathbf{X} \sim U(-1, 1)$  and  $\mathbf{Y} \triangleq \alpha \tan(\pi\mathbf{X}/2)$ . Show (a) that  $\mathbf{Y}$  is a Cauchy random variable, and (b) find the density function of  $\mathbf{Z} \triangleq 1/\mathbf{X}^2$ . (This defines  $\mathbf{Z}$  except when  $\mathbf{X} = 0$ , so to complete the definition, when  $\mathbf{X} = 0$  make  $\mathbf{Z} = 0$  to be definite, but it won't change the answer if we gave it any other value since  $\mathbf{X} = 0$  is an almost null event.)
7. A student number at the University of Ottawa is a string of 7 or 9 digits in a row. A random experiment consists of picking at random a digit from one such student number—your student number. Suppose we then define a random variable  $\mathbf{X}$  to correspond to the value of the chosen digit. Find the mean and variance of that random variable.
8. Find the mean and variance of the random variable  $\mathbf{X}$  having the distribution given in Question 1 on the previous assignment.
9. If  $\mathbf{X}$  has a mean of 2 and variance of 5, find the mean and variance of  $\mathbf{Y} \triangleq 3\mathbf{X} - 1$ .
10. The mark of a randomly selected student in a certain course is found to follow a certain distribution with mean  $m$  and standard deviation  $\sigma$ . The marks are to be adjusted following the mapping where the original mark  $x$  is changed to  $y = ax + b$ . What should  $a$  and  $b$  be if the new marks are to have a mean  $m'$  and standard deviation  $\sigma'$ ? There are two answers to this question, algebraically, but only one is appropriate for an application. Explain why.



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SOLUTIONS

1/ Let us first solve the problem first from first principles:

$$F_{\mathbf{Y}}(y) = \mathcal{P}(\mathbf{Y} \leq y) = \mathcal{P}(2\mathbf{X} + 3 \leq y) = \mathcal{P}(\mathbf{X} \leq \frac{1}{2}(y - 3)) = F_{\mathbf{X}}((y - 3)/2)$$

$$\implies f_{\mathbf{Y}}(y) = \frac{1}{2}f_{\mathbf{X}}((y - 3)/2).$$

Now  $f_{\mathbf{X}}(x) = \begin{cases} \frac{1}{6}, & -2 < x < 4; \\ 0, & \text{otherwise,} \end{cases}$  so we get that

$$f_{\mathbf{Y}}(y) = \begin{cases} \frac{1}{12}, & -1 < y < 11; \\ 0, & \text{otherwise.} \end{cases} \implies \mathbf{X} \sim U(-1, 11)$$

*Alternative approach:*  $\mathbf{Y} = g(\mathbf{X})$  for  $g(x) = 2x + 3$ ,  $g'(x) = 2 (\neq 0)$ , and  $\mathbf{X}$  is a continuous random variable so  $\mathbf{Y}$  must also be a continuous random variable. The equation  $y = g(x)$  has exactly one solution for each value of  $y$  which is  $x = (y - 3)/2$ , so  $f_{\mathbf{Y}}(y) = \frac{1}{2}f_{\mathbf{X}}((y - 3)/2)$  which is the same answer as above.

3/  $\mathbf{X} \sim U(-2, 4)$ ,  $\mathbf{Y} = \mathbf{X}^2 u(\mathbf{X}) = \begin{cases} \mathbf{X}^2, & \text{if } \mathbf{X} > 0; \\ 0, & \mathbf{X} \leq 0. \end{cases}$

The event  $\{\mathbf{Y} \leq y\} = \begin{cases} \{\mathbf{X}^2 u(\mathbf{X}) \leq y\}, & \text{if } y \geq 0; \\ \emptyset, & \text{if } y < 0; \end{cases} = \begin{cases} \{\mathbf{X} \leq \sqrt{y}\}, & \text{if } y \geq 0; \\ \emptyset, & \text{if } y < 0. \end{cases} \implies F_{\mathbf{Y}}(y) = \underbrace{\begin{cases} F_{\mathbf{X}}(\sqrt{y}), & \text{if } y \geq 0; \\ 0, & \text{if } y < 0. \end{cases}}_{\text{has a jump at } y = 0}$

Thus  $f_{\mathbf{Y}}(y) = F_{\mathbf{X}}(0)\delta(y) + \begin{cases} \frac{f_{\mathbf{X}}(\sqrt{y})}{2\sqrt{y}}, & \text{if } y > 0; \\ 0, & \text{otherwise.} \end{cases}$  As  $F_{\mathbf{X}}(0) = \mathcal{P}(\mathbf{X} \leq 0) = \frac{1}{3}$  and  $f_{\mathbf{X}}(x) = \begin{cases} \frac{1}{6}, & \text{if } -2 < x < 4; \\ 0, & \text{otherwise.} \end{cases}$ ,

$$f_{\mathbf{Y}}(y) = \frac{1}{3}\delta(y) + \begin{cases} \frac{1}{12\sqrt{y}}, & 0 < y < 16; \\ 0, & \text{otherwise.} \end{cases}$$

4/  $\mathbf{X} \sim N(3, 4)$ ,

$$g(x) = \begin{cases} x, & x \geq 0; \\ x^3, & x < 0. \end{cases}$$

$$\mathbf{Y} = g(\mathbf{X}) = \begin{cases} \mathbf{X}, & \text{if } \mathbf{X} > 0; \\ \mathbf{X}^3, & \mathbf{X} \leq 0. \end{cases}$$

The event  $\{\mathbf{Y} \leq y\} = \begin{cases} \{\mathbf{X} \leq y\}, & \text{if } y \geq 0; \\ \{\mathbf{X}^3 \leq y\}, & \text{if } y < 0; \end{cases} = \begin{cases} \{\mathbf{X} \leq y\}, & \text{if } y \geq 0; \\ \{\mathbf{X} \leq \sqrt[3]{y}\}, & \text{if } y < 0; \end{cases} \implies F_{\mathbf{Y}}(y) = \begin{cases} F_{\mathbf{X}}(y), & \text{if } y \geq 0; \\ F_{\mathbf{X}}(\sqrt[3]{y}), & \text{if } y < 0. \end{cases}$   
 has no jump at  $y = 0$

Thus  $f_{\mathbf{Y}}(y) = \begin{cases} f_{\mathbf{X}}(y), & \text{if } y \geq 0; \\ \frac{1}{3\sqrt[3]{y^2}} f_{\mathbf{X}}(\sqrt[3]{y}), & \text{if } y < 0. \end{cases}$  But we have that  $f_{\mathbf{X}}(x) = \frac{1}{\sqrt{8\pi}} e^{-(x-3)^2/8}$ , so

$$f_{\mathbf{Y}}(y) = \frac{1}{\sqrt{8\pi}} e^{-(y-3)^2/8} u(y) + \frac{1}{\sqrt{72\pi} \sqrt[3]{y^2}} e^{-(\sqrt[3]{y}-3)^2/8} u(-y).$$

5/  $\mathbf{Y} = g(\mathbf{X})$  for  $g(x) = e^x \implies g'(x) = e^x = g(x)$ . The equation  $y = g(x)$  has no solution if  $y \leq 0$  and only one solution  $x = \ln y$  when  $y > 0$ . At the one solution point  $g'(x) = e^{+\ln y} = y$ , so

$$f_{\mathbf{Y}}(y) = \begin{cases} \frac{1}{y} f_{\mathbf{X}}(\ln y), & y > 0; \\ 0, & y \leq 0; \end{cases} = \begin{cases} \frac{2}{y^3}, & y > 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then from this  $\mathcal{E}\{\mathbf{Y}\} = \int_{-\infty}^{\infty} y f_{\mathbf{Y}}(y) dy = \int_1^{\infty} 2y^{-2} dy = -2y^{-1} \Big|_1^{\infty} = 2$ .

[As a check on this, the other way we have to find  $\mathcal{E}\{g(\mathbf{X})\}$  is

$$\mathcal{E}\{g(\mathbf{X})\} = \int_{-\infty}^{\infty} g(x) f_{\mathbf{X}}(x) dx = \int_0^{\infty} 2e^{-2x} e^x dx = \int_0^{\infty} 2e^{-x} dx = -2e^{-x} \Big|_0^{\infty} = 2. ]$$

6/ (a)  $\mathbf{Y} \triangleq \alpha \tan(\pi \mathbf{X}/2)$ . The solutions to the equation  $y = \alpha \tan(\pi x/2)$  are the elements of the infinite collection of points  $x_n = \frac{2}{\pi} \tan^{-1}(y/\alpha) + 2n$  where  $n \in \mathbb{Z}$ . Since the function  $\tan^{-1}(x)$  has a value in  $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ , the only solution point in the  $[-1, 1]$  range is that for  $n = 0$ . Also, for  $g(x) = \alpha \tan(\pi x/2)$ ,

$$g'(x) = \frac{1}{2} \alpha \pi \sec^2(\pi x/2) = \frac{1}{2} \alpha \pi [1 + \tan^2(\pi x/2)].$$

At the solution points,  $g'(x) = \frac{1}{2} \alpha \pi [1 + (y/\alpha)^2] = \frac{\pi}{2\alpha} (\alpha^2 + y^2)$ . Thus

$$f_{\mathbf{Y}}(y) = \sum_{n=-\infty}^{\infty} \frac{f_{\mathbf{X}}(\frac{2}{\pi} \tan^{-1}(y/\alpha) + 2n)}{\underbrace{\frac{\pi}{2\alpha} (\alpha^2 + y^2)}_{\substack{\text{is zero except for } n = 0 \\ \text{when the numerator is } \frac{1}{2}}}} = \frac{\alpha}{\pi(\alpha^2 + y^2)}.$$

(b)  $\mathbf{Z} \triangleq 1/|\mathbf{X}|^3$ . The equation  $z = g(x) \triangleq 1/|x|^3$  has no solutions if  $z \leq 0$  and two solutions  $x_1 = 1/\sqrt[3]{z}$  and  $x_2 = -1/\sqrt[3]{z}$  when  $z > 0$ . Then  $g'(x) = -3/x^4 \text{sgn}(x)$  which has an absolute value of  $|g'(x)| = 3/x^4$  which is  $3z^{4/3}$  at both solution points. Thus

$$f_{\mathbf{Z}}(z) = \begin{cases} \frac{f_{\mathbf{X}}(1/\sqrt[3]{z}) + f_{\mathbf{X}}(-1/\sqrt[3]{z})}{3z^{4/3}}, & \text{if } z > 0; \\ 0, & \text{otherwise;} \end{cases} = \begin{cases} \frac{1}{3} z^{-4/3}, & \text{if } z > 1; \\ 0, & \text{otherwise.} \end{cases}$$

7/ The answer to this question depends on your student number, but it is easy to find these answers. First, let  $\mathbf{Y}$  be a random variable that corresponds to the position in the student number that is picked.  $\mathbf{Y}$  is equally likely to be any integer from 1 to 7 if your student number has seven digits, and is equally likely to be any integer from 1 to 9 if the student number has nine digits. Let  $g(n)$  be a function on  $\{1, 2, 3, \dots, m\}$ , where  $m$  is the number of digits in the student number, that gives the value of the  $n$ th digit. Then  $\mathbf{X} = g(\mathbf{Y})$  and so

$$\mathcal{E}\{\mathbf{X}\} = \mathcal{E}\{g(\mathbf{Y})\} = \left\{ \frac{1}{7} \sum_{n=1}^7 g(n) \text{ or } \frac{1}{9} \sum_{n=1}^9 g(n) \right\} = \text{average of the digits in the student number}$$

$$\mathcal{E}\{\mathbf{X}^2\} = \mathcal{E}\{g^2(\mathbf{Y})\} = \left\{ \frac{1}{7} \sum_{n=1}^7 g^2(n) \text{ or } \frac{1}{9} \sum_{n=1}^9 g^2(n) \right\} = \text{average of the square of the digits in the student number}$$

$$\text{var}(\mathbf{X}) = \mathcal{E}\{\mathbf{X}^2\} - [\mathcal{E}\{\mathbf{X}\}]^2$$

For the fictitious student number 9103219,  $\mathcal{E}\{\mathbf{X}\} = \frac{25}{7} \simeq 3.5714$ ,  $\mathcal{E}\{\mathbf{X}^2\} = \frac{177}{7} \simeq 25.286 \implies \text{var}(\mathbf{X}) = \frac{613}{49} \simeq 12.531$ . Each student will have a different answer.

8/ 
$$f_{\mathbf{X}}(x) = \begin{cases} 3x^2, & \text{if } 0 \leq x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\mathcal{E}\{\mathbf{X}\} = \int_0^1 3x^3 dx = \frac{3}{4}x^4 \Big|_0^1 = \frac{3}{4} = 0.75.$$

$$\mathcal{E}\{\mathbf{X}^2\} = \int_0^1 3x^4 dx = \frac{3}{5}x^5 \Big|_0^1 = \frac{3}{5} = 0.60.$$

It then follows that  $\text{var}(\mathbf{X}) = \mathcal{E}\{\mathbf{X}^2\} - [\mathcal{E}\{\mathbf{X}\}]^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{80} \simeq 0.0375$ .

9/  $\mathcal{E}\{\mathbf{Y}\} = \mathcal{E}\{3\mathbf{X} - 1\} = 3\mathcal{E}\{\mathbf{X}\} - 1 = 3 \times 2 - 1 = 5$ .

In general,  $\text{var}(\alpha\mathbf{X} + \beta) = \mathcal{E}\{[\alpha\mathbf{X} + \beta - (\alpha\mathcal{E}\{\mathbf{X}\} + \beta)]^2\} = \mathcal{E}\{\alpha^2[\mathbf{X} - \mathcal{E}\{\mathbf{X}\}]^2\} = \alpha^2\text{var}(\mathbf{X})$ . Thus, specifically, here ( $\alpha = 3$ ,  $\beta = -1$ ),  $\text{var}(\mathbf{Y}) = 3^2\text{var}(\mathbf{X}) = 45$ .

10/ Let  $\mathbf{X}$  be the mark of the student and  $\mathbf{X}'$  be the adjusted mark. Then  $\mathbf{X}' = a\mathbf{X} + b$ , and so

$$\begin{aligned} \mathcal{E}\{\mathbf{X}'\} &= a\mathcal{E}\{\mathbf{X}\} + b = am + b = m' & \implies & a = \frac{\sigma'}{\sigma}, \quad b = m' - \left(\frac{\sigma'}{\sigma}\right)m. \\ \text{var}(\mathbf{X}') &= a^2\text{var}(\mathbf{X}) = a^2\sigma^2 = (\sigma')^2 \end{aligned}$$

Note: Algebraically, there is another solution to the above equations which is  $a = -\sigma'/\sigma$ ,  $b = m' + (\sigma'/\sigma)m$ , but this solution would not be acceptable in this scenario since it would mean that better performance would lead to lower scores—you are giving weaker results the higher marks and stronger results the lower marks. We clearly want that  $a > 0$ .