Université d'Ottawa
Faculté de génie

École de science informatique
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L'Université canadienne
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University of Ottawa
Faculty of Engineering

School of Electrical Engineering
and Computer Science

ELG 3126 **RANDOM SIGNALS AND SYSTEMS** **Winter 2023**

ASSIGNMENT 7

(due at 8:30 AM Thursday, March 9 in class)

1. Determine the exact value of $\mathcal{P}(\mathbf{X} \geq c)$ as a function of c and the bound on the probability provided by the Markov inequality in the cases that (a) $\mathbf{X}$ is a random variable uniformly distributed on the interval $[0, b]$, and (b) $\mathbf{X}$ is a an Rayleigh random variable, and (c) $\mathbf{X}$ is a discrete random variable equally likely to be any natural number value from 1 to N. Compare.

2. Determine the exact value of $\mathcal{P}(|\mathbf{X} - \mathcal{E}\{\mathbf{X}\}| > c)$ as a function of c and the bound on the probability provided by the Chebyshev inequality in the cases that (a) $\mathbf{X}$ is a random variable uniformly distributed on the interval $[-b, b]$, and (b) $\mathbf{X}$ is a Laplacian random variable with parameter α. Compare.

3. From the text: 4.102.

4. We know that if $\mathbf{X}$ is a real random variable with a mean value and α is a real number, then $\mathcal{E}\{\alpha\mathbf{X}\} = \alpha\mathcal{E}\{\mathbf{X}\}$, Show that if $\mathbf{Z}$ is a complex random variable with a mean, then for any complex value β, $\mathcal{E}\{\beta\mathbf{Z}\} = \beta\mathcal{E}\{\mathbf{Z}\}$. Hint Remember that in class we defined the mean of a complex random variable $\mathbf{W}$ to be $\mathcal{E}\{\mathbf{W}\} \triangleq \mathcal{E}\{\mathrm{Re}[\mathbf{W}]\} + j\mathcal{E}\{\mathrm{Im}[\mathbf{W}]\}$.

5. A discrete random variable $\mathbf{X}$ can have values -1, 0, and 1, each with equal probability. Find the characteristic function of $\mathbf{X}$ and from this, use the moment theorem to find the moments of the random variable.

6. Find the characteristic function of $\mathbf{Y} = a\mathbf{X} + b$ in terms of the characteristic function of $\mathbf{X}$, and then use that and the moment theorem to show that the mean of $\mathbf{Y}$ is $a\mathcal{E}\{\mathbf{X}\} + b$ provided $\mathbf{X}$ has a mean.

7. Show that the characteristic function of a Cauchy random variable $\mathbf{X}$ is $\Phi_{\mathbf{X}}(\omega) = e^{-\alpha|\omega|}$. What does the moment theorem say about the mean value of $\mathbf{X}$?
   Hint: $\alpha^2 + \omega^2 = (\alpha + j\omega)(\alpha - j\omega)$.

8. From the text: 5.2. Assume each coin flip is an independent experiment.

9. For a pair of random variables $\mathbf{X}$ and $\mathbf{Y}$, each of the events below can be expressed in the form $(\mathbf{X}, \mathbf{Y}) \in \mathcal{I}$ for some region $\mathcal{I}$ in $\mathbb{R}^2$. Sketch $\mathcal{I}$ for each case.
   (a) $3\mathbf{X} - 5\mathbf{Y} > 10$, (b) $\min\{\mathbf{X}, \mathbf{Y}\} > 1$, (c) $\mathbf{X}/\mathbf{Y} < 0$, (d) $\mathbf{X}^2\mathbf{Y}^2 \leq 1$, (e) $\max\{|\mathbf{X}|, |\mathbf{Y}|\} < 1$.

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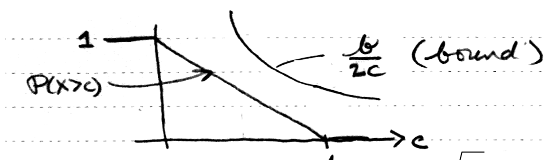
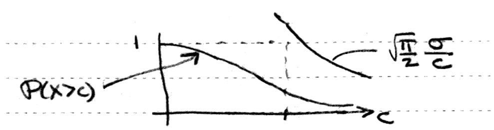
ASSIGNMENT 7

SOLUTIONS

1/ (a) If $\mathbf{X} \sim U(0,b)$, then $\mathcal{E}\{\mathbf{X}\} = \frac{1}{2}b$ and $\mathcal{P}(\mathbf{X} \geq c) = \begin{cases} 1, & \text{if } c \leq 0; \\ 1 - \dfrac{c}{b}, & \text{if } 0 < c < b; \\ 0, & \text{if } c \geq b. \end{cases}$ The Markov inequality states that for $c > 0$, $\mathcal{P}(\mathbf{X} \geq c) < \dfrac{b}{2c}$.



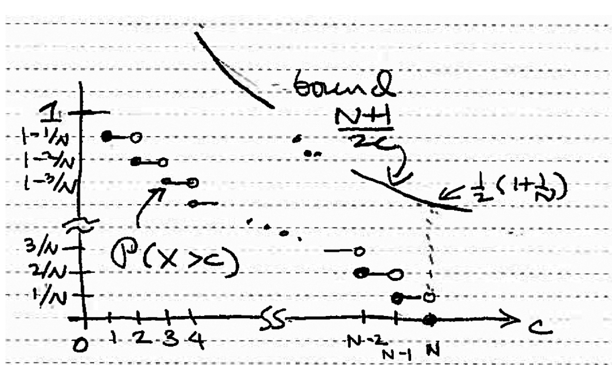
(b) If $f_{\mathbf{X}}(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} u(x)$, then $\mathcal{E}\{\mathbf{X}\} = \sqrt{\dfrac{\pi}{2}}\sigma$, and $\mathcal{P}(\mathbf{X} \geq c) = e^{-c^2/2\sigma^2}$. The Markov inequality states that for $c > 0$, $\mathcal{P}(\mathbf{X} \geq c) = e^{-c^2/2\sigma^2} < \sqrt{\dfrac{\pi}{2}}\dfrac{\sigma}{c}$.



(c) If $\mathbf{X}$ is equally likely to be any value from 1 to N, then $\mathcal{E}\{\mathbf{X}\} = \frac{1}{2}(N+1)$ and

$$\mathcal{P}(\mathbf{X} \geq c) = \begin{cases} 1, & \text{if } c \leq 0; \\ 1 - \frac{\lfloor c \rfloor}{N}, & \text{if } 0 < c < N; \\ 0, & \text{if } c > N, \end{cases}$$

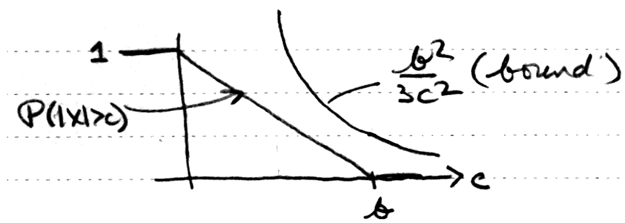
where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. The Markov inequality states that for $c > 0$, $\mathcal{P}(\mathbf{X} \geq c) < \dfrac{(N+1)}{2c}$.



. . . *(over)*

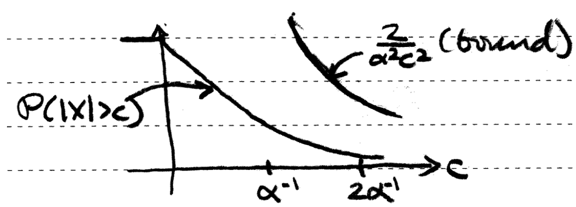
2/ (a) If $\mathbf{X} \sim U(-b,b)$, then $\mathcal{E}\{\mathbf{X}\}=0$, $\text{var}(\mathbf{X})=b^2/3$, and $\mathcal{P}(|\mathbf{X}-\mathcal{E}\{\mathbf{X}\}|\geq c)=\mathcal{P}(|\mathbf{X}|\geq c)=\begin{cases}0, & \text{if } c\geq b;\\ 1-\dfrac{c}{b}, & \text{if } 0<c<b;\\ 1, & \text{if } c\leq 0.\end{cases}$

The Chebyshev inequality states that for $c>0$, $\mathcal{P}(|\mathbf{X}|\geq c)<\dfrac{b^2}{3c^2}$.



(b) If $f_{\mathbf{X}}(x)=\frac{1}{2}\alpha e^{-\alpha|x|}$, then $\mathcal{E}\{\mathbf{X}\}=0$, $\text{var}(\mathbf{X})=2/\alpha^2$, and $\mathcal{P}(|\mathbf{X}-\mathcal{E}\{\mathbf{X}\}|\geq c)=\mathcal{P}(|\mathbf{X}|\geq c)=\begin{cases}e^{-\alpha c}, & \text{if } c>0;\\ 1, & \text{if } c\leq 0.\end{cases}$

The Chebyshev inequality states that for $c>0$, $\mathcal{P}(|\mathbf{X}|\geq c)<\dfrac{2}{\alpha^2 c^2}$.



3/ (4.102)

(a)
$$\Phi_{\mathbf{X}}(\omega)=\int_{-b}^{b}\frac{1}{2b}e^{j\omega x}\,dx=\left.\frac{e^{j\omega x}}{2j\omega b}\right|_{-b}^{b}=\text{sinc}(b\,\omega/\pi).$$

(b)
$$\Phi_{\mathbf{X}}^{(1)}(\omega)\triangleq\frac{d}{d\omega}\Phi_{\mathbf{X}}(\omega)=\frac{d}{d\omega}\frac{\sin(b\,\omega)}{b\,\omega}=\frac{b\,\omega\cos(b\,\omega)-\sin(b\,\omega)}{b\,\omega^2}\quad\text{if }\omega\neq 0;\quad\Longrightarrow\ \Phi_{\mathbf{X}}^{(1)}(0)=0\Longrightarrow\ \mathcal{E}\{\mathbf{X}\}=0.$$

$$\Phi_{\mathbf{X}}^{(2)}(\omega)\triangleq\frac{d^2}{d\omega^2}\Phi_{\mathbf{X}}(\omega)=\frac{d}{d\omega}\frac{b\,\omega\cos(b\,\omega)-\sin(b\,\omega)}{b\,\omega^2}=\frac{[2-b^2\omega^2]\sin(b\,\omega)-2b\,\omega b\,\omega\cos(b\,\omega)}{b\,\omega^3}\quad\text{if }\omega\neq 0.$$

To get the value at $\omega=0$ we find the limit of the above expression as $\omega\to 0$ applying l'Hôpital's rule:

$$\Phi_{\mathbf{X}}^{(2)}(0)=\lim_{\omega\to 0}\frac{-b^2\cos(b\,\omega)}{3}=-\tfrac{1}{3}b^2\quad\Longrightarrow\quad\mathcal{E}\{\mathbf{X}^2\}=-(j^2)\tfrac{1}{3}b^2=\tfrac{1}{3}b^2.$$

Thus $\text{var}(\mathbf{X})=\mathcal{E}\{\mathbf{X}^2\}-\mathcal{E}^2\{\mathbf{X}\}=\frac{1}{3}b^2$.

4/ From the definition of the expectation, we observe

$$\begin{aligned}\mathcal{E}\{\beta\mathbf{Z}\}&=\mathcal{E}\{\text{Re}[\beta\mathbf{Z}]\}+j\mathcal{E}\{\text{Im}[\beta\mathbf{Z}]\}\\&=\mathcal{E}\{\text{Re}[\beta]\text{Re}[\mathbf{Z}]-\text{Im}[\beta]\text{Im}[\mathbf{Z}]\}+j\mathcal{E}\{\text{Re}[\beta]\text{Im}[\mathbf{Z}]+\text{Im}[\beta]\text{Re}[\mathbf{Z}]\}\\&=\text{Re}[\beta]\mathcal{E}\{\text{Re}[\mathbf{Z}]\}-\text{Im}[\beta]\mathcal{E}\{\text{Im}[\mathbf{Z}]\}+j\text{Re}[\beta]\mathcal{E}\{\text{Im}[\mathbf{Z}]\}+j\text{Im}[\beta]\mathcal{E}\{\text{Re}[\mathbf{Z}]\}\\&=[\text{Re}[\beta]+j\text{Im}[\beta]]\mathcal{E}\{\text{Re}[\mathbf{Z}]\}+j[\text{Re}[\beta]+j\text{Im}[\beta]]\mathcal{E}\{\text{Im}[\mathbf{Z}]\}\\&=\beta[\mathcal{E}\{\text{Re}[\mathbf{Z}]\}+j\mathcal{E}\{\text{Im}[\mathbf{Z}]\}]\\&=\beta\mathcal{E}\{\mathbf{Z}\}\end{aligned}$$

as claimed.

. . . *(over)*

5/ $\Phi_{\mathbf{X}}(\omega) = \mathcal{E}\{e^{j\omega\mathbf{X}}\} = \frac{1}{3}[e^{j\omega(-2)} + e^{-j0} + e^{j2\omega}] = \frac{1}{3}[1 + 2\cos(2\omega)] = \frac{1}{3} + \frac{2}{3}\cos(2\omega)$. Then

$$\Phi_{\mathbf{X}}^{(n)}(\omega) = \begin{cases} -\frac{2}{3}2^n \sin(\omega), & \text{if } n = 1, 5, 9, \ldots; \\ -\frac{2}{3}2^n \cos(\omega), & \text{if } n = 2, 6, 10, \ldots; \\ \frac{2}{3}2^n \sin(\omega), & \text{if } n = 3, 7, 11, \ldots; \\ \frac{2}{3}2^n \cos(\omega), & \text{if } n = 4, 8, 12, \ldots; \end{cases} \implies \Phi_{\mathbf{X}}^{(n)}(0) = \begin{cases} 0, & \text{if } n = 1, 5, 9, \ldots; \\ -\frac{1}{3}2^{n+1}, & \text{if } n = 2, 6, 10, \ldots; \\ 0, & \text{if } n = 3, 7, 11, \ldots; \\ \frac{1}{3}2^{n+1}, & \text{if } n = 4, 8, 12, \ldots. \end{cases}$$

The Moment Theorem gives us, for $n \geq 0$,

$$\mathcal{E}\{\mathbf{X}^n\} = (-j)^n \Phi_{\mathbf{X}}^{(n)}(0) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n = 0 \\ \frac{1}{3}2^{n+1}, & \text{if } n = 2, 4, 6, \ldots; \end{cases}$$

6/ $\Phi_{\mathbf{Y}}(\omega) = \mathcal{E}\{e^{j\omega\mathbf{Y}}\} = \mathcal{E}\{e^{j\omega[a\mathbf{X}+b]}\} = \mathcal{E}\{e^{j\omega b}e^{ja\omega\mathbf{X}}\} = e^{j\omega b}\mathcal{E}\{e^{ja\omega\mathbf{X}}\} = e^{j\omega b}\Phi_{\mathbf{x}}(a\omega)$,

Now by the product rule, $\Phi'_{\mathbf{Y}}(\omega) = ae^{j\omega b}\Phi'_{\mathbf{x}}(a\omega) + (jb)e^{j\omega b}\Phi_{\mathbf{x}}(a\omega)$, so $\Phi'_{\mathbf{Y}}(\omega) = a\Phi'_{\mathbf{x}}(a\omega) + jb$. This gives us that $\Phi'_{\mathbf{Y}}(0) = aj\mathcal{E}\{\mathbf{X}\} + jb$, which gives us $\mathcal{E}\{\mathbf{Y}\} = -j\Phi'_{\mathbf{Y}}(0) = a\mathcal{E}\{\mathbf{X}\} + b$ as required.

7/ The characteristic function of $\mathbf{X}$ at $-\omega$ is $\mathcal{F}\{f_{\mathbf{X}}(x)\} = \mathcal{F}\{\frac{\alpha}{\pi}\frac{1}{\alpha^2 + x^2}\}$. But

$$\frac{\alpha}{\pi}\frac{1}{\alpha^2 + x^2} = \frac{\alpha}{\pi}\frac{1}{(\alpha + jx)(\alpha - jx)} = \frac{1}{2\pi}[\frac{1}{\alpha + jx} + \frac{1}{\alpha - jx}]$$

We know from Fourier analysis that the Fourier transform of $h(t) = e^{-\alpha t}u(t)$ is $H(f) = \dfrac{1}{\alpha + j\omega}$ if $\alpha > 0$, so by duality, the Fourier transform of $H(x)$ is $2\pi h(-\omega)$.giving us that the Fourier transform of $\dfrac{1}{(\alpha + jx)}$ is $2\pi e^{\alpha\omega}u(-\omega)$, and the Fourier transform of $\dfrac{1}{(\alpha - jx)}$ is $2\pi e^{-\alpha\omega}u(\omega)$. This allows us to conclude that that the the Fourier transform of $f_{\mathbf{X}}(x)$ is $e^{-\alpha\omega}u(\omega) + e^{\alpha\omega}u(-\omega) = e^{-\alpha|\omega|}$, and so the characteristic function of $\mathbf{X}$ is

$$\Phi_{\mathbf{X}}(\omega) = e^{-\alpha|-\omega|} = e^{-\alpha|\omega|}.$$

The Moment Theorem states that $\mathcal{E}\{\mathbf{X}\} = -j\Phi'_{\mathbf{X}}(0)$, but here $\Phi'_{\mathbf{X}}(\omega)$ does not exist at $\omega = 0$ (a corner point of $\Phi_{\mathbf{X}}(\omega)$), which agrees with the result that $\mathcal{E}\{\mathbf{X}\}$ does not exist.

8/ (5.2)

(a) Each person's outcome can be described as an ordered pair of toss results, so for both persons we can have a pair of such pairs giving outcomes as $(\underbrace{(H,H)}_{\text{Carlos}}, \underbrace{(H,T)}_{\text{Michael}})$ simplified as (HH, HT):

$$\mathcal{S} = \{(HH,HH),(HH,HT),(HH,TH),(HH,TT),(HT,HH),(HT,HT),(HT,TH),(HT,TT) \\ (TH,HH),(TH,HT),(TH,TH),(TH,TT),(TT,HH),(TT,HT),(TT,TH),(TT,TT)\}$$

The values for $(\mathbf{X},\mathbf{Y})$ for each point in $\mathcal{S}$ listed in the order above are respectively

$$(0,4),(1,3),(1,3),(2,2),(1,3),(0,2),(0,2),(1,1),(1,3),(0,2),(0,2),(1,1),(2,2),(1,1),(1,1),(0,0)$$

There are 6 possible values for $(\mathbf{X},\mathbf{Y})$: $(0,0),(0,2),(0,4),(1,1),(1,3),(2,2)$ (the range of $(\mathbf{X},\mathbf{Y})$).

. . . *(over)*

8/ (continued)

(b) All the singletons for the samplespace in (a) are equiprobably, so

$$\begin{aligned}
\mathcal{P}(\mathbf{X}=0,\mathbf{Y}=0) &= \mathcal{P}(\{(TT,TT)\}) = \tfrac{1}{16};\\
\mathcal{P}(\mathbf{X}=0,\mathbf{Y}=2) &= \mathcal{P}(\{(HT,HT),(HT,TH),(TH,HT),(TH,TH)\}) = \tfrac{1}{4};\\
\mathcal{P}(\mathbf{X}=0,\mathbf{Y}=4) &= \mathcal{P}(\{(HH,HH)\}) = \tfrac{1}{16};\\
\mathcal{P}(\mathbf{X}=1,\mathbf{Y}=1) &= \mathcal{P}(\{(HT,TT),(TH,TT),(TT,HT),(TT,TH)\}) = \tfrac{1}{4};\\
\mathcal{P}(\mathbf{X}=1,\mathbf{Y}=3) &= \mathcal{P}(\{(HH,HT),(HH,TH),(HT,HH),(TH,HH)\}) = \tfrac{1}{4};\\
\mathcal{P}(\mathbf{X}=2,\mathbf{Y}=2) &= \mathcal{P}(\{(HH,TT),(TT,HH)\}) = \tfrac{1}{8}.
\end{aligned}$$

(c) $\mathcal{P}(\mathbf{X}+\mathbf{Y}=1) = \mathcal{P}(\varnothing) = 0.$,
$\mathcal{P}(\mathbf{X}+\mathbf{Y}=2) = \mathcal{P}(\{(HT,HT),(HT,TH),(TH,HT),(TH,TH),(HT,TT),(TH,TT),(TT,HT),(TT,TH)\}) = \tfrac{1}{2}.$

9/ For (a), $\mathcal{I}$ is the region to the right and below the line $3x-5y=10$; for (b) we note that $\min\{x,y\}>1$ is equivalent to requiring $x>1$ and $y>1$; for (c) we see simply that x and y must be nonzero and have different signs; for (d) we see that the condition $x^3y^2 \le 1$ is equivalent to the condition $|xy| \le 1$ and thus is the region bounded the two hyperbolas $xy=1$ and $xy=-1$; and lastly, for (e) we note that $\max\{|x|,|y|\} < 1$ is equivalent to requiring $|x|<1$ and $|y|<1$: