Université d'Ottawa
Faculté de génie

École de science informatique
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L'Université canadienne
Canada's university

University of Ottawa
Faculty of Engineering

School of Electrical Engineering
and Computer Science

ELG 3126 **RANDOM SIGNALS AND SYSTEMS** **Winter 2023**

ASSIGNMENT 8

(due at 8:30 AM Thursday, March 16 in class)

1. A point is selected from the triangle on the Cartesian plane defined by $\{(x, y) \mid 0 < x < y < 1\}$. Assume that all points in the triangle are equally likely. Let $\mathbf{X}$ be defined to the the x-coordinate of the selected point and
   (a) Find the joint cdf of $\mathbf{X}$ and $\mathbf{Y}$.
   (b) Find the marginal cdf of $\mathbf{X}$ and of $\mathbf{Y}$.
   (c) Find in terms of the joint cdf the probability of the events (i) $\mathcal{A} = \{\mathbf{X} \le \frac{1}{4}\}$; (ii) $\mathcal{B} = \{\frac{1}{4} < \mathbf{X} \le \frac{3}{4}, \frac{1}{4} < \mathbf{Y} \le \frac{3}{4}\}$.
   (d) Find the joint pdf of $\mathbf{X}$ and $\mathbf{Y}$.
   (e) Find the marginal pdf of $\mathbf{X}$ and of $\mathbf{Y}$.
   (f) Find $\mathcal{P}(\mathbf{Y} < \mathbf{X}^2)$.

2. Can the function
$$F(x, y) = \begin{cases} 1 - e^{-(x^2+y)}, & x \ge 0, \, y \ge 0; \\ 0, & \text{otherwise}; \end{cases}$$
   be the joint cdf of some two random variables $\mathbf{X}$ and $\mathbf{Y}$. If it can be, what would $F_{\mathbf{X}}(x)$ then be? *Hint*: If two random variables $\mathbf{X}$ and $\mathbf{Y}$ had joint probability distribution functions $F_{\mathbf{XY}}(x, y) = F(x, y)$, what would $\mathcal{P}(0 < \mathbf{X} \le 1, 0 < \mathbf{Y} \le 1)$ be?

3. From the text: 5.18, 5.32, 5.42. To avoid any misinterpretation, take the target center to be at the origin, and let us explicitly define $\boldsymbol{\theta} \triangleq \arg(\mathbf{X}_1 + j\mathbf{X}_2)$ which is a value in the $(-\pi, \pi]$ range.

4. Two jointly continuous random variables $\mathbf{X}$ and $\mathbf{Y}$ have a joint density function
$$f_{\mathbf{XY}}(x, y) = \begin{cases} ky^2(1 - x^2), & \text{if } 0 < x < 1 \text{ and } 0 < y < 1; \\ 0, & \text{otherwise}. \end{cases}$$
   where k is a constant.
   (a) Find k.
   (b) Find $F_{\mathbf{XY}}(x, y)$.
   (c) Find $f_{\mathbf{X}}(x)$ and $f_{\mathbf{Y}}(y)$.
   (d) Find $\mathcal{P}(\mathbf{X} < \mathbf{Y})$, $\mathcal{P}(\mathbf{Y} < \mathbf{X}^2)$, and $\mathcal{P}(\mathbf{X} + \mathbf{Y} > \frac{1}{2})$.
   (e) Determine if $\mathbf{X}$ and $\mathbf{Y}$ are independent?
   (f) Find $f_{\mathbf{X}}(x \mid \mathbf{Y} = y)$ for all y (including negative values for y).

. . . *(over)*

5. Two jointly continuous random variables $\mathbf{X}$ and $\mathbf{Y}$ have a joint density function

$$f_{\mathbf{XY}}(x,y) = \begin{cases} k[\cos(x+y)+\cos(x-y)], & \text{if } 0 < x < \frac{1}{2}\pi \text{ and } 0 < y < \frac{1}{2}\pi; \\ 0, & \text{otherwise.} \end{cases}$$

where k is a constant.

(a) Find k.

(b) Find $F_{\mathbf{XY}}(x,y)$.

(c) Find $f_{\mathbf{X}}(x)$ and $f_{\mathbf{Y}}(y)$.

(d) Find $\mathcal{P}(\mathbf{X} < \mathbf{Y})$ and $\mathcal{P}(\mathbf{X}+\mathbf{Y} > \frac{1}{2}\pi)$.

(e) Determine if $\mathbf{X}$ and $\mathbf{Y}$ are independent?

(f) Find $f_{\mathbf{X}}(x \mid \mathbf{Y} = y)$ for all $y \in \mathbb{R}$.

6. Two jointly continuous random variables $\mathbf{X}$ and $\mathbf{Y}$ have a joint density function

$$f_{\mathbf{XY}}(x,y) = \begin{cases} k(x+y+xy), & \text{if } 0 < x < 1 \text{ and } 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

where k is a constant.

(a) Find k.

(b) Find $F_{\mathbf{XY}}(x,y)$.

(c) Find $f_{\mathbf{X}}(x)$ and $f_{\mathbf{Y}}(y)$.

(d) Find $\mathcal{P}(\mathbf{X} < \mathbf{Y})$, $\mathcal{P}(\mathbf{Y} < \mathbf{X} + \frac{1}{2})$, and $\mathcal{P}(\mathbf{X}+\mathbf{Y} > 1)$.

(e) Determine if $\mathbf{X}$ and $\mathbf{Y}$ are independent?

7. A point is chosen at random from the region on the Cartesian plane $\{(x,y) \mid |x| + 2|y| \le 2\}$. The location of the point in Cartesian coordinates is $(\mathbf{X}, \mathbf{Y})$. Find the joint pdf $f_{\mathbf{XY}}(x,y)$, and the marginal density functions $f_{\mathbf{X}}(x)$ and $f_{\mathbf{Y}}(y)$, and determine if $\mathbf{X}$ and $\mathbf{Y}$ are independent random variables.

8. Suppose $\mathbf{X}$ and $\mathbf{Y}$ are two independent random variables both uniformly distributed on $[0,2]$. Find (i) $\mathcal{P}(\mathbf{X} < \mathbf{Y}^3)$, (ii) $\mathcal{P}(\mathbf{X}+\mathbf{Y} < \frac{1}{2})$ and (iii) $\mathcal{P}(\min(\mathbf{X},\mathbf{Y}) > \frac{1}{2})$.

9. Find the joint cdf of $\mathbf{Z} = \min(\mathbf{X},\mathbf{Y})$ and $\mathbf{W} = \max(\mathbf{X},\mathbf{Y})$ when $\mathbf{X}$ and $\mathbf{Y}$ are independent and $\mathbf{X} \sim N(0,1)$, $\mathbf{Y} \sim N(0,1)$.

ELG 3126 **RANDOM SIGNALS AND SYSTEMS** **Winter 2022**

ASSIGNMENT 8

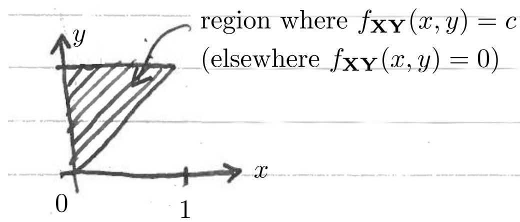
SOLUTIONS

1/ If a density function $f_{\mathbf{XY}}(x,y) = c$ (a constant) for all $(x,y) \in \mathcal{D}$ and zero otherwise, then

$$1 = \iint_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)\,dxdy = c \times \text{area of } \mathcal{D}, \qquad \Longrightarrow \qquad c = 1/\text{area of } \mathcal{D}.$$

Then for any (reasonable) set $\mathcal{A}$. $\mathcal{P}((\mathbf{X},\mathbf{Y}) \in \mathcal{A}) = \dfrac{\text{area of } \mathcal{A} \cap \mathcal{D}}{\text{area of } \mathcal{D}}$.

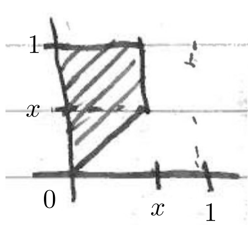
In this case the region $\mathcal{D}$ is a triangle with area $\frac{1}{2}$, so $c = 2$: $f_{\mathbf{XY}}(x,y) = \begin{cases} 2 & \text{if } 0 < x < y < 1; \\ 0, & \text{otherwise,} \end{cases}$ and the probability of $(\mathbf{X},\mathbf{Y}) \in \mathcal{A}$ for some (reasonable) set $\mathcal{A}$ contained in $\mathcal{D}$ is $2 \times$ area of $\mathcal{A}$



(a) $F_{\mathbf{XY}}(x,y) = \mathcal{P}(\mathbf{X} \le x, \mathbf{Y} \le y)$.

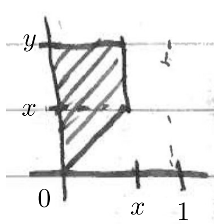
If $x \le 0$ or $y \le 0$ this is clearly 0, and for $x \ge 1$ *and* $y \ge 1$, this is 1.

For $x \in (0,1)$ and $y \ge 1$ this is the probability of picking a point in the region depicted below:



$$\text{area} = x - \tfrac{1}{2}x^2 \quad \Longrightarrow \quad F_{\mathbf{XY}}(x,y) = 2x - x^2.$$

For $x \in (0,1)$ and $x \le y < 1$ this is the probability of picking a point in the region depicted below:

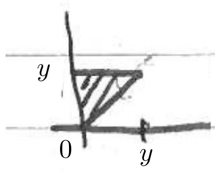


$$\text{area} = xy - \tfrac{1}{2}x^2 \quad \Longrightarrow \quad F_{\mathbf{XY}}(x,y) = 2xy - x^2.$$

. . . *(continued on the next page)*

1/ (a) *(continued)*

For $x > y$ and $y \in (0,1)$, this is the probability of picking a point in the region depicted below:



area $= \frac{1}{2}y^2 \implies F_{\mathbf{XY}}(x,y) = y^2$.

Hence the overall result is

$$F_{\mathbf{XY}}(x,y) - \begin{cases} 1, & x \geq 1,\ y \geq 1; \\ y^2, & x > y,\ 0 < y < 1; \\ 2xy - x^2 & 0 < x \leq y < 1; \\ 2x - x^2, & 0 < x < 1,\ y \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

(b) $F_{\mathbf{X}}(x) = F_{\mathbf{XY}}(x,\infty) = \begin{cases} 1, & x \geq 1; \\ 2x - x^2, & 0 < x < 1; \\ 0, & x \leq 0. \end{cases}$

$F_{\mathbf{Y}}(y) = F_{\mathbf{XY}}(\infty,y) = \begin{cases} 1, & y \geq 1; \\ y^2, & 0 < y < 1; \\ 0, & y \leq 0. \end{cases}$

(c) $\mathcal{P}(\mathcal{A}) = F_{\mathbf{XY}}(\frac{1}{4},\infty) = \frac{1}{2} - (\frac{1}{4})^2 = \frac{7}{16}$;
$\mathcal{P}(\mathcal{B}) = F_{\mathbf{XY}}(\frac{1}{4},\frac{1}{4}) + F_{\mathbf{XY}}(\frac{3}{4},\frac{3}{4}) - F_{\mathbf{XY}}(\frac{3}{4},\frac{1}{4}) - F_{\mathbf{XY}}(\frac{1}{4},\frac{3}{4}) = \frac{1}{8} + \frac{9}{16} - \frac{1}{16} - \frac{5}{16} = \frac{5}{16}$.

(d) $f_{\mathbf{XY}}(x,y) = \dfrac{\partial^2}{\partial x \partial y} F_{\mathbf{XY}}(x,y) = \begin{cases} 2 & \text{if } 0 < x < y < 1; \\ 0, & \text{otherwise;} \end{cases}$ (which if course agrees with what was stated at the start).

(e) $f_{\mathbf{X}}(x) = \dfrac{d}{dx} F_{\mathbf{X}}(x) = \begin{cases} 2 - 2x, & 0 < x < 1. \\ 0, & \text{otherwise.} \end{cases}$

$f_{\mathbf{Y}}(y) = \dfrac{d}{dy} F_{\mathbf{Y}}(y) = \begin{cases} 2y, & 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases}$

(f) The region in the triangle over which $f_{\mathbf{XY}}(x,y)$ is nonzero where $y < x^2$ is the empty set as in the triangle $y > x > x^2$ (as $x \in [0,1]$). Hence the needed probability is 0.

2/ We know from the Existence Theorem for the joint cdf that $F(x,y)$ is a joint cdf of some pair of random variables if and only if it has the five properties

1. $F(\infty,\infty) = 1$;
2. $F(-\infty,y) = F(x,-\infty) = F(-\infty,-\infty) = 0$ for all x, y;
3. If $x_1 < x_2$ and $y_1 < y_2$, then $F(x_1,y_1) \leq F(x_2,y_2)$;
4. $F(x+0,y+0) = F(x,y)$;
5. If $x_1 < x_2$ and $y_1 < y_2$, then $F(x_1,y_1) + F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1) \geq 0$.

It is obvious that the given function satisfies the first four of these properties, so the question of whether $F(x,y)$ is the joint density function of some random variable pair depends on the last property. This in turns depends on whether the function $\frac{\partial^2}{\partial x \partial y} F(x,y) > 0$ for all x, y since

$$F(x_1,y_1) + F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{\partial^2}{\partial x \partial y} F(x,y)\, dy\, dx.$$

Here for $x > 0, y > 0$, $\frac{\partial^2}{\partial x \partial y} F(x,y) = -2xe^{-(x^2+y)}$ which is decidedly negative in all cases. Hence, $F(x,y)$ cannot be the joint cdf of some pair of random variables.

. . . *(continued on the next page)*

2/ *(continued)*

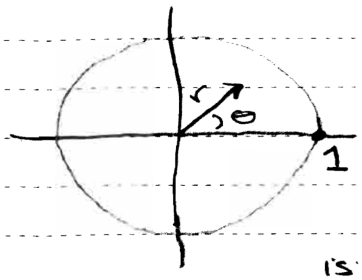
To arrive at this conclusion more simply, if $\mathbf{X}$ and $\mathbf{Y}$ were two real random variables for which $F_{\mathbf{XY}}(x,y) = F(x,y)$, then as we know $\mathcal{P}(0 < \mathbf{X} \leq 1, 0 < \mathbf{Y} \leq 1) = F_{\mathbf{XY}}(0,0) + F_{\mathbf{XY}}(1,1) - F_{\mathbf{XY}}(0,1) - F_{\mathbf{XY}}(1,0)$, we would have

$$\mathcal{P}(0 < \mathbf{X} \leq 1, 0 < \mathbf{Y} \leq 1) = F(0,0) + F(1,1) - F(0,1) - F(1,0) = 2e^{-1} - e^{-2} - 1 \simeq -0.39958 < 0,$$

which violates the fifth property.

Remark We could have proceeded more simply to say that if $F(x,y)$ was a joint cdf, then $\frac{\partial^2}{\partial x \partial y} F(x,y)$ must be the corresponding joint pdf, and we see that $\frac{\partial^2}{\partial x \partial y} F(x,y)$ does not have the properties of a joint pdf (specifically of being nonnegative).

3/ (5.18)

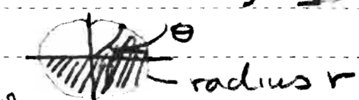


$R = \sqrt{X^2+Y^2}$ $\quad \underline{\Theta} = \underbrace{\arg(X+jY)}_{\text{a value in } (-\pi, \pi]}$

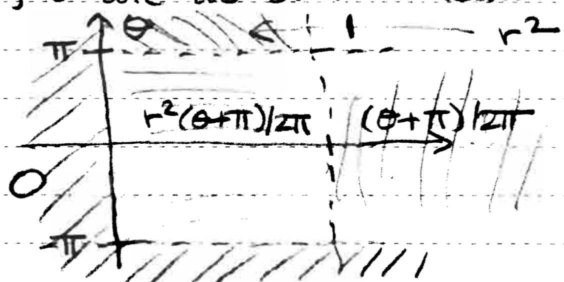
($R \geq 0$ always)

(a) The event $R \leq r$, $\underline{\Theta} \leq \theta$ when $r \in [0,1]$, $\theta \in (-\pi, \pi]$

is the event (X,Y) is in the region



which has area $\pi r^2 \frac{(\theta+\pi)}{2\pi}$ $\Rightarrow$ probability of landing in this region is $r^2(\theta+\pi)/2\pi$. If $r > 1$ or $\theta > \pi$, the region is the same as if $r=1$ or $\theta = \pi$, while if $r < 0$ or $\theta \leq -\pi$ the region is the empty set. Thus the cdf of $R, \underline{\Theta}$ for all values r, θ are as shown below



(b) $F_R(r) = F_{R\underline{\Theta}}(r, \infty) = \begin{cases} 1 & r \geq 1 \\ r^2 & 0 < r < 1 \\ 0 & r \leq 0 \end{cases}$

$F_{\underline{\Theta}}(\theta) = F_{R\underline{\Theta}}(\infty, \theta) = \begin{cases} 1 & \theta \geq \pi \\ (\theta+\pi)/2\pi & |\theta| < \pi \\ 0 & \theta \leq -\pi \end{cases}$

(c) $\mathcal{P}(R > \frac{1}{2}, 0 < \underline{\Theta} < \frac{\pi}{2}) = \mathcal{P}(\frac{1}{2} < R \leq 1, 0 < \underline{\Theta} < \frac{\pi}{2})$

$= F_{R\underline{\Theta}}(\frac{1}{2}, 0) + F_{R\underline{\Theta}}(1, \pi/2) - F_{R\underline{\Theta}}(\frac{1}{2}, \frac{\pi}{2}) - F_{R\underline{\Theta}}(1, 0)$

$= \frac{1}{8} + \frac{3}{4} - \frac{3}{16} - \frac{1}{2} = \underline{\underline{\frac{3}{16}}}$

. . . *(problem continued on the next page)*

3/ (continued)

(5.32) (a) $f_{R\Theta}(r,\theta) = \frac{\partial^2}{\partial r \partial \theta} F_{R\Theta}(r,\theta) = \begin{cases} r/\pi, & 0<r<1, -\pi<\theta<\pi; \\ 0, & \text{otherwise} \end{cases}$

(b) $f_R(r) = \frac{d}{dr} F_R(r) = \begin{cases} 2r, & 0<r<1 \\ 0, & \text{otherwise} \end{cases}$

$f_\Theta(\theta) = \frac{d}{d\theta} F_\Theta(\theta) = \begin{cases} 1/2\pi, & -\pi<\theta<\pi; \\ 0, & \text{otherwise.} \end{cases}$

(5.42) $f_R(r) f_\Theta(\theta) = f_{R\Theta}(r,\theta)$ so R & Θ are independent.

4/ (a) $1 = \iint\limits_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)\,dx\,dy = \int_0^1\int_0^1 ky^2(1-x^2)\,dx\,dy = \frac{2}{3}k\int_0^1 y^2\,dy = \frac{2}{9}k \implies k = \frac{9}{2} = 4.5$.

(b) $$F_{\mathbf{XY}}(x,y) = \int_{-\infty}^{x}(\int_{-\infty}^{y} f_{\mathbf{XY}}(\alpha,\beta)\,d\beta)\,d\alpha = \begin{cases} 1, & x \ge 1,\ y \ge 1; \\ \frac{3}{2}x - \frac{1}{2}x^3, & 0 \le x < 1,\ y > 1; \\ y^3, & x > 1,\ 0 \le y < 1; \\ \frac{3}{2}y^3(x - \frac{1}{3}x^3), & 0 \le x < 1,\ 0 \le y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(c) $$f_{\mathbf{X}}(x) = \int_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)\,dy = \begin{cases} \frac{3}{2}(1-x^2), & 0 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$f_{\mathbf{Y}}(y) = \int_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)\,dx = \begin{cases} 3y^2, & 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(d) $$\mathcal{P}(\mathbf{X} < \mathbf{Y}) = \int_0^1 [\int_0^y \tfrac{9}{2}y^2(1-x^2)\,dx\,]\,dy = \int_0^1 \tfrac{9}{2}y^2[y - \tfrac{1}{3}y^3]\,dy = \int_0^1 \tfrac{9}{2}y^3 - \tfrac{3}{2}y^5\,dy = \tfrac{9}{8} - \tfrac{1}{4} = \tfrac{7}{8}.$$

$$\mathcal{P}(\mathbf{Y} < \mathbf{X}^2) = \int_0^1 [\int_0^{x^2} \tfrac{9}{2}y^2(1-x^2)\,dy\,]\,dx = \int_0^1 \tfrac{3}{2}(1-x^2)x^6\,dx = \tfrac{3}{2}\int_0^1 x^6 - x^8\,dx = \tfrac{3}{2}(\tfrac{1}{7} - \tfrac{1}{9}) = \tfrac{1}{21}.$$

$$\mathcal{P}(\mathbf{X} + \mathbf{Y} > \tfrac{1}{2}) = 1 - \mathcal{P}(\mathbf{X} + \mathbf{Y} \le \tfrac{1}{2}) = 1 - \int_0^{1/2} [\int_0^{(1/2)-x} \tfrac{9}{2}y^2(1-x^2)\,dy\,]\,dx = 1 - \int_0^{1/2} \tfrac{3}{2}(1-x^2)(\tfrac{1}{2} - x)^3\,dx$$

$$= 1 - \tfrac{3}{16}\int_0^{1/2} 1 - 6x + 11x^2 - 2x^3 - 12x^4 + 8x^5\,dx$$

$$= 1 - \tfrac{3}{512}(16 - 24 + \tfrac{44}{3} - 1 - \tfrac{12}{5} + \tfrac{2}{3}) = \tfrac{2501}{2560} \simeq 0.97695$$

(e) Since $f_{\mathbf{X}}(x)f_{\mathbf{Y}}(y) = \begin{cases} \frac{9}{2}y^2(1-x^2), & \text{if } 0 < x < 1 \text{ and } 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases} = f_{\mathbf{XY}}(x,y)$ for all $x, y \in \mathbb{R}$, we see that $\mathbf{X}$ and $\mathbf{Y}$ are independent.

(f) Since $\mathbf{X}$ and $\mathbf{Y}$ are independent, $f_{\mathbf{X}}(x \mid \mathbf{Y} = y) = f_{\mathbf{X}}(x)$ whenever $f_{\mathbf{Y}}(y) > 0$, and is undefined when $f_{\mathbf{Y}}(y) = 0$:

$$f_{\mathbf{X}}(x \mid \mathbf{Y} = y) = \begin{cases} \frac{3}{2}(1-x^2), & 0 < x < 1, 0 < y < 1; \\ 0, & (x \le 0 \text{ or } x \ge 1), 0 < y < 1; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

5/ First let us note that since $\cos(x+y)+\cos(x-y)=2\cos(x)\cos(y)$,

$$f_{\mathbf{XY}}(x,y)=\begin{cases}2k\cos(x)\cos(y), & \text{if } 0<x<\frac{1}{2}\pi \text{ and } 0<y<\frac{1}{2}\pi;\\ 0, & \text{otherwise.}\end{cases}$$

(a) $1=\iint\limits_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)\,dx\,dy=\int\limits_0^{\frac{\pi}{2}}\int\limits_0^{\frac{\pi}{2}} 2k\cos(x)\cos(y)\,dx\,dy=2k\int_0^{\frac{\pi}{2}}\cos(y)\,dy=2k \implies k=\frac{1}{2}.$

(b) $$F_{\mathbf{XY}}(x,y)=\int_{-\infty}^{x}\Big(\int_{-\infty}^{y} f_{\mathbf{XY}}(\alpha,\beta)\,d\beta\Big)\,d\alpha=\begin{cases}1, & x\ge\frac{\pi}{2},\ y\ge\frac{\pi}{2};\\ \sin(x), & 0\le x<\frac{\pi}{2},\ y>1;\\ \sin(y), & x>1,\ 0\le y<\frac{\pi}{2};\\ \sin(x)\sin(y), & 0\le x<\frac{\pi}{2},\ 0\le y<\frac{\pi}{2};\\ 0, & \text{otherwise.}\end{cases}$$

(c) $$f_{\mathbf{X}}(x)=\int_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)\,dy=\begin{cases}\cos(x), & 0<x<\frac{\pi}{2};\\ 0, & \text{otherwise.}\end{cases}$$

$$f_{\mathbf{Y}}(y)=\int_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)\,dx=\begin{cases}\cos(y), & 0<y<\frac{\pi}{2};\\ 0, & \text{otherwise.}\end{cases}$$

(d) $$\mathcal{P}(\mathbf{X}<\mathbf{Y})=\int\limits_0^{\frac{\pi}{2}}\Big[\int\limits_0^{y}\cos(x)\cos(y)\,dx\Big]\,dy=\int_0^{\frac{\pi}{2}}\sin(y)\cos(y)\,dy=\int_0^{\frac{\pi}{2}}\tfrac{1}{2}\sin(2y)\,dy=-\tfrac{1}{4}[\cos(\pi)-\cos(0)]=\tfrac{1}{2}.$$

$$\mathcal{P}(\mathbf{X}+\mathbf{Y}>\tfrac{1}{2}\pi)=1-\mathcal{P}(\mathbf{X}+\mathbf{Y}\le\tfrac{1}{2}\pi)=1-\int\limits_0^{\frac{\pi}{2}}\Big[\int\limits_0^{\frac{\pi}{2}-x}\cos(x)\cos(y)\,dy\Big]\,dx=1-\int\limits_0^{\frac{\pi}{2}}\cos(x)\sin(\tfrac{\pi}{2}-x)\,dx$$

$$=1-\int_0^{\frac{\pi}{2}}\cos^2(x)\,dx=1-\tfrac{1}{2}\int_0^{\frac{\pi}{2}}1+\cos(2x)\,dx$$

$$=1-\tfrac{\pi}{4}\simeq 0.21460$$

(e) Since clearly $f_{\mathbf{X}}(x)f_{\mathbf{Y}}(y)=f_{\mathbf{XY}}(x,y)$ for all $x,y\in\mathbb{R}$, we see that $\mathbf{X}$ and $\mathbf{Y}$ are independent.

(f) Since $\mathbf{X}$ and $\mathbf{Y}$ are independent, $f_{\mathbf{X}}(x\mid\mathbf{Y}=y)=f_{\mathbf{X}}(x)$ whenever $f_{\mathbf{Y}}(y)>0$, and is undefined when $f_{\mathbf{Y}}(y)=0$:

$$f_{\mathbf{X}}(x\mid\mathbf{Y}=y)=\begin{cases}\cos(x), & 0<x<\frac{\pi}{2},\ 0<y<\frac{\pi}{2};\\ 0, & (x\le 0 \text{ or } x\ge 1),\ 0<y<\frac{\pi}{2};\\ \text{undefined}, & \text{otherwise.}\end{cases}$$

6/ (a) $1=\iint\limits_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)\,dx\,dy=\int\limits_0^1\int\limits_0^1 k(x+y+xy)\,dx\,dy=\frac{1}{2}k\int_0^1 1+3y\,dy=\frac{5}{4}k \implies k=\frac{4}{5}=0.8$.

(b) $$F_{\mathbf{XY}}(x,y)=\int_{-\infty}^{x}\Big(\int_{-\infty}^{y} f_{\mathbf{XY}}(\alpha,\beta)\,d\beta\Big)\,d\alpha=\begin{cases}1, & x\ge 1,\ y\ge 1;\\ \frac{1}{5}(3x^2+2x), & 0\le x<1,\ y>1;\\ \frac{1}{5}(3y^2+2y), & x>1,\ 0\le y<1;\\ \frac{1}{5}(2x^2y+2y^2x+x^2y^2), & 0\le x<1,\ 0\le y<1;\\ 0, & \text{otherwise.}\end{cases}$$

(c) $$f_{\mathbf{X}}(x)=\int_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)\,dy=\begin{cases}\frac{2}{5}(3x+1), & 0<x<1;\\ 0, & \text{otherwise.}\end{cases}$$

$$f_{\mathbf{Y}}(y)=\int_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)\,dx=\begin{cases}\frac{2}{5}(3y+1), & 0<y<1;\\ 0, & \text{otherwise.}\end{cases}$$

. . . (problem continued on the next page)

6/ (continued)

(d) $\mathcal{P}(\mathbf{X} < \mathbf{Y}) = k \int_0^1 [\int_0^y x + y + xy\, dx\,]\, dy = \frac{2}{5} \int_0^1 3y^2 + y^3\, dy = \frac{1}{2}.$

$\mathcal{P}(\mathbf{Y} < \mathbf{X} + \frac{1}{2}) = 1 - \mathcal{P}(\mathbf{Y} \geq \mathbf{X} + \frac{1}{2}) = 1 - k \int_0^{1/2} [\int_{x+1/2}^1 x + y + xy\, dy\,]\, dx = 1 - \frac{1}{8}k \int_0^{1/2} 3 + 3x - 16x^2 - 4x^3\, dx$

$= 1 - \frac{11}{96} = \frac{85}{96} \simeq 0.88542.$

$\mathcal{P}(\mathbf{X} + \mathbf{Y} > 1) = 1 - \mathcal{P}(\mathbf{X} + \mathbf{Y} \leq 1) = 1 - k \int_0^1 [\int_0^{1-x} x + y + xy\, dy\,]\, dx = 1 - \frac{1}{2}k \int_0^1 1 + x - 3x^2 + x^3\, dx$

$= 1 - \frac{3}{10} = \frac{7}{10}.$

(e) Since $f_{\mathbf{X}}(x) f_{\mathbf{Y}}(y) = \begin{cases} \frac{4}{25}(3x+1)(3y+1), & \text{if } 0 < x < 1 \text{ and } 0 < y < 1; \\ 0, & \text{otherwise;} \end{cases}$, which is not $f_{\mathbf{XY}}(x, y)$ for all $x, y \in \mathbb{R}$, we see that $\mathbf{X}$ and $\mathbf{Y}$ are not independent.

7/ Clearly the joint pdf is a function that is constant for all (x, y) inside the diamond shapped area with corners at (0,1), (2,0), $(0, -1)$ and $(-2, 0)$, whose area is half that of a 4 units by 2 units rectangle, so since the area of this diamond is 4,

$$f_{\mathbf{XY}}(x, y) = \begin{cases} \frac{1}{4}, & \text{if } |x| + 2|y| < 2; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$f_{\mathbf{X}}(x) = \int_{-\infty}^{\infty} f_{\mathbf{XY}}(x, y)\, dy = \begin{cases} \int_{-(1-\frac{1}{2}|x|)}^{1-\frac{1}{2}|x|} \frac{1}{4}\, dy = \frac{1}{4}(2 - |x|), & \text{if } |x| < 2; \\ 0, & \text{otherwise.} \end{cases}$$

Similarly,

$$f_{\mathbf{Y}}(y) = \begin{cases} 1 - |y|, & \text{if } |y| < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Obviously $f_{\mathbf{XY}}(x, y) \neq f_{\mathbf{X}}(x) f_{\mathbf{Y}}(y)$ for some $x, y \in \mathbb{R}$ (e.g., for $x = 1$, $y = \frac{1}{4}$, the LHS is $\frac{1}{4}$, while the RHS is $\frac{1}{4} \times \frac{3}{8} = \frac{3}{32}$), so $\mathbf{X}$ and $\mathbf{Y}$ are not independent.

8/
$$f_{\mathbf{XY}}(x, y) = \begin{cases} \frac{1}{4}, & \text{if } 0 < x < 2, 0 < y < 2; \\ 0, & \text{otherwise.} \end{cases},$$

(i) $\mathcal{P}(\mathbf{X} < \mathbf{Y}^3) = 1 - \mathcal{P}(\mathbf{X} \geq \mathbf{Y}^3) = 1 - \int_0^2 [\int_0^{\sqrt[3]{x}} \frac{1}{4}\, dy]\, dx = 1 - \int_0^2 \frac{1}{4} x^{1/3}\, dx = 1 - \frac{3}{8}\sqrt[3]{2} \simeq 0.52753.$

(ii) $\mathcal{P}(\mathbf{X} + \mathbf{Y} < \frac{1}{2})$ is the probability that $(\mathbf{X}, \mathbf{Y})$ lies in the portion of the half plane that lies below the line where $x + y = \frac{1}{2}$, that is in the $[0, 2] \times [0, 2]$ square. This is a triangular region with vertices at (0,0), $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. The area of the triangle is $\frac{1}{8}$, so the probability is $\frac{1}{32} = 0.03125$.

(iii) $\mathcal{P}(\min(\mathbf{X}, \mathbf{Y}) > \frac{1}{2})$ is the probability that $\mathbf{X} > \frac{1}{2}$ and $\mathbf{Y} > \frac{1}{2}$, which is, by virtue of the independence of $\mathbf{X}$ and $\mathbf{Y}$, is $\mathcal{P}(\mathbf{X} > \frac{1}{2})\mathcal{P}(\mathbf{Y} > \frac{1}{2}) = (\frac{3}{4})^2 = \frac{9}{16} = 0.5625.$

9/ Since each random variable has an $N(0,1)$ distribution

$$f_{\mathbf{X}}(x) = f_{\mathbf{Y}}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \Longrightarrow \qquad F_{\mathbf{X}}(x) = 1 - Q(x) = F_{\mathbf{Y}}(x).$$

Now $\mathbf{Z} = \min(\mathbf{X}.\mathbf{Y})$, $\mathbf{W} = \max(\mathbf{X}.\mathbf{Y})$, so

$$\begin{aligned}
F_{\mathbf{ZW}}(z,w) &= \mathcal{P}(\underbrace{\min(\mathbf{X},\mathbf{Y}) \leq z}_{\{\mathbf{X} \leq z\} \cup \{\mathbf{Y} \leq z\}}, \underbrace{\max(\mathbf{X}.\mathbf{Y}) \leq w}_{\{\mathbf{X} \leq w\} \cap \{\mathbf{Y} \leq w\}}) \\
&= \mathcal{P}([\mathbf{X} \leq z, \mathbf{X} \leq w, \mathbf{Y} \leq w] \cup [\mathbf{X} \leq w, \mathbf{Y} \leq z, \mathbf{Y} \leq w]) \\
&= \mathcal{P}(\mathbf{X} \leq z, \mathbf{X} \leq w, \mathbf{Y} \leq w) + \mathcal{P}(\mathbf{X} \leq w, \mathbf{Y} \leq z, \mathbf{Y} \leq w) - \mathcal{P}(\mathbf{X} \leq z, \mathbf{X} \leq w, \mathbf{Y} \leq z, \mathbf{Y} \leq w) \\
&= \mathcal{P}(\mathbf{X} \leq \min(z,w))\mathcal{P}(\mathbf{Y} \leq w) + \mathcal{P}(\mathbf{X} \leq w)\mathcal{P}(\mathbf{Y} \leq \min(z,w)) - \mathcal{P}(\mathbf{X} \leq \min(z,w))\mathcal{P}(\mathbf{Y} \leq \min(z,w)) \\
&= F_{\mathbf{X}}(\min(z,w))F_{\mathbf{Y}}(w) + F_{\mathbf{X}}(w)F_{\mathbf{Y}}(\min(z,w)) - F_{\mathbf{X}}(\min(z,w))F_{\mathbf{Y}}(\min(z,w)).
\end{aligned}$$

Then introducing the actual joint cdf of the random variables we get

$$F_{\mathbf{ZW}}(z,w) = 2[1 - Q(\min(z,w))][1 - Q(w)] - [1 - Q(\min(z,w))]^2 = [1 - Q(\min(z,w))][1 - 2Q(w) + Q(\min(z,w))].$$

Expanding further gives us

$$F_{\mathbf{ZW}}(z,w) = \begin{cases} [1 - Q(w)]^2, & \text{if } w < z; \\ [1 - Q(z)][1 + Q(z) - 2Q(w)], & \text{if } z \leq w. \end{cases}$$

Although it was not asked, we can now easily find the joint density function by differentiating $F_{\mathbf{ZW}}(z,w)$:

$$f_{\mathbf{ZW}}(z,w) = \begin{cases} \frac{1}{\pi} e^{-(z^2+w^2)/2}], & \text{if } z \leq w; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\dfrac{d}{dx}Q(x) = -\dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

ASSIGNMENT 8— In Class Component

1. Let $\mathbf{X}$, $\mathbf{Y}$ be two random variables with joint density function

$$f_{\mathbf{XY}}(x,y) = \begin{cases} K(3x^2+2y+6xy^2), & \text{if } 0<x<2,\ 0<y<2; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find K.

(b) Find $\mathcal{P}(\mathbf{X} > \frac{1}{2}, \mathbf{Y} > 1)$.

(c) Find $\mathcal{P}(\mathbf{XY} > 1)$.

(d) Find the marginal density function of $\mathbf{X}$ and the marginal density function of $\mathbf{Y}$. Are $\mathbf{X}$ and $\mathbf{Y}$ independant?

Solution:

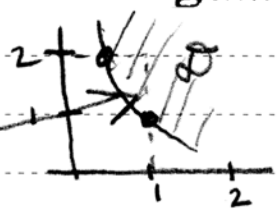
(a) $1 = \int_{-\infty}^{\infty}\int f_{XY}(x,y)\,dx\,dy = K\int_0^2\left(\int_0^2 3x^2+2y+6xy^2\,dy\right)dx$

$= K\int_0^2 6x^2+4+16x\,dx = K(16+8+32) = 56K \Rightarrow K = \frac{1}{56}$

(b) $P(X>1/2, Y>1) = \int_{1/2}^{2}\left(\int_1^2 K(3x^2+2y+6xy^2)\,dy\right)dx = K\int_{1/2}^{2} 3x^2+3+14x\,dx$

$= K\cdot\left(\frac{63}{8}+\frac{9}{2}+\frac{14}{2}\cdot\frac{15}{4}\right) = K\left(\frac{63+36+210}{8}\right) = \frac{309}{448} \simeq 0.6897$

(c) $P(XY>1) = P((X,Y)\in D)$



$= \int_{1/2}^{2}\left(\int_{1/x}^{2} f_{XY}(x,y)\,dy\right)dx$

$= K\cdot\int_{1/2}^{2} \underbrace{3x^2(2-\tfrac{1}{x}) + (4-\tfrac{1}{x^2}) + 2x(8-1/x^3)}_{6x^2+13x+4-\frac{3}{x^2}}\,dx = \frac{333}{448} \simeq 0.7433$

(d) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \begin{cases}(3x^2+8x+2)/28, & 0<x\le 2 \\ 0, & \text{otherwise}\end{cases}$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \begin{cases}(3y^2+y+2)/14 & 0<y\le 2 \\ 0, & \text{otherwise}\end{cases}$

Thus $f_{\mathbf{X}}(x)f_{\mathbf{Y}}(y) \neq f_{\mathbf{XY}}(x,y)$ for almost all x, y, giving us that $\mathbf{X}$ and $\mathbf{Y}$ are *not* independent.