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**ELG 3126** **RANDOM SIGNALS AND SYSTEMS** **Winter 2023**

# ASSIGNMENT 9

(due at 8:30 AM Thursday, March 23 in class)

1. The random variables $\mathbf{X}$ and $\mathbf{Y}$ have a joint density function given by
$$f_{\mathbf{XY}}(x,y) = \begin{cases} \frac{2}{\pi} e^{-(x^2+y^2)}, & \text{if } xy > 0; \\ 0, & \text{otherwise;} \end{cases}$$
Find the joint density function of $\mathbf{Z} = \mathbf{XY}$ and $\mathbf{W} = \mathbf{X}/\mathbf{Y}$ and the marginal density function of $\mathbf{W}$. Are $\mathbf{Z}$ and $\mathbf{W}$ independent?
*Hint*: You don't need to explicitly find the density functions for both $\mathbf{Z}$ and $\mathbf{W}$, but just consider $f_{\mathbf{Z}}(z \mid \mathbf{W} = w)$.

2. Suppose $\mathbf{X}$ and $\mathbf{Y}$ have the joint density function
$$f_{\mathbf{XY}}(x,y) = \begin{cases} (y+1)e^{-(x+y+xy)}, & x > 0 \text{ and } y > 0; \\ 0, & \text{otherwise,} \end{cases}$$
Find the joint density function of $\mathbf{Z} = \mathbf{X}^2\mathbf{Y}^2$ and $\mathbf{W} = \mathbf{Y}^2/\mathbf{X}^2$.

3. The random variables $\mathbf{X}$ and $\mathbf{Y}$ two independent $U(0,1)$ random variables. Use the technique of an auxiliary random variable to find the density function of $\mathbf{Z} = \mathbf{XY}$.

4. Suppose that $\mathbf{X}$, $\mathbf{Y}$ are two random variables with means 2 and -3 (respectively), variance 3 and 4 (respectively) and covariance -1.
   (a) What is the correlation coefficient of these two random variables?
   (b) What is the correlation matrix of these two random variables?
   (c) What is the covariance matrix of these two random variables?
   (d) What is the mean, mean power and variance of the random variable $\mathbf{Z} = \mathbf{X} - 2\mathbf{Y}$.

5. Suppose $\boldsymbol{\theta} \sim U(0,\pi)$, and $\mathbf{X} = \cos^2(\boldsymbol{\theta}/2)$, $\mathbf{Y} = \sin^2(\boldsymbol{\theta}/2)$. Determine if $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated.

6. For two real random variables $\mathbf{X}$ and $\mathbf{Y}$ that have finite second order moments, it is easy to see that for all real values $\alpha$, $\mathcal{E}\{[\mathbf{X}+\alpha\mathbf{Y}]^2\} \geq 0$, and that
$$\mathcal{E}\{[\mathbf{X}+\alpha\mathbf{Y}]^2\} = \mathcal{E}\{\mathbf{X}^2 + \alpha^2\mathbf{Y}^2 + 2\alpha\mathbf{XY}\} = \alpha^2\mathcal{E}\{\mathbf{Y}^2\} + 2\alpha\mathcal{E}\{\mathbf{XY}\} + \mathcal{E}\{\mathbf{X}^2\}.$$
   (a) Find the value of $\alpha$ that minimizes $\mathcal{E}\{[\mathbf{X}+\alpha\mathbf{Y}]^2\}$, expressing it in terms of the correlation between $\mathbf{X}$ and $\mathbf{Y}$, $\text{cor}(\mathbf{X},\mathbf{Y})$, $\mathcal{E}\{\mathbf{X}^2\}$ and $\mathcal{E}\{\mathbf{Y}^2\}$. Assume that $\mathcal{E}\{\mathbf{Y}^2\} > 0$.
   (b) Using the fact that the minimum value of $\mathcal{E}\{[\mathbf{X}+\alpha\mathbf{Y}]^2\}$ must be non-negative, show that
$$[\mathcal{E}\{\mathbf{XY}\}]^2 \leq \mathcal{E}\{\mathbf{X}^2\}\mathcal{E}\{\mathbf{Y}^2\} \qquad (\dagger)$$
   when $\mathcal{E}\{\mathbf{Y}^2\} > 0$. Argue that we can drop the assumption that $\mathcal{E}\{\mathbf{Y}^2\} > 0$ so that (†) holds for all $\mathbf{X}$ and $\mathbf{Y}$ which both have a finite mean square value. This is a very important result that is known as the *Cauchy-Schwarz inequality* (for random variables). Under what conditions will we have equality in this expression?
   *Hint*: If $\mathcal{E}\{\mathbf{Y}^2\} = 0$ but $\mathcal{E}\{\mathbf{X}^2\} > 0$, interchanging $\mathbf{X}$ and $\mathbf{Y}$ gives us that (†) holds in that case, leaving only to show (†) holds when $\mathcal{E}\{\mathbf{X}^2\} = \mathcal{E}\{\mathbf{Y}^2\} = 0$. For that just consider $\mathcal{E}\{[\mathbf{X}+\mathbf{Y}]^2\} \geq 0$ and $\mathcal{E}\{[\mathbf{X}-\mathbf{Y}]^2\} \geq 0$ to show that if $\mathcal{E}\{\mathbf{X}^2\} = \mathcal{E}\{\mathbf{Y}^2\} = 0$, then $\mathcal{E}\{\mathbf{XY}\} = 0$ so (†) holds then as well (with equality).

6. [*continued*]

(c) Based on (b), show that

$$[\text{cov}(\mathbf{X}, \mathbf{Y})]^2 \leq \text{var}(\mathbf{X})\text{var}(\mathbf{Y}),$$

and then that $[\rho_{\mathbf{XY}}]^2 \leq 1$.

7. Suppose that **X**, **Y** are two random variables with means 1 and -1 (respectively) and covariance matrix

$$\mathbf{C}_{\mathbf{XY}} = \begin{bmatrix} 2 & -1 \\ -1 & 9 \end{bmatrix}$$

Find the mean, correlation matrix and covariance matrix of $\mathbf{Z} = \mathbf{X} + 2\mathbf{Y} - 3$, $\mathbf{W} = 3\mathbf{X} - 7\mathbf{Y} + 12$.

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# ASSIGNMENT 9

## SOLUTIONS

1/ First we solve the simultaneous equations $z = xy$, $w = x/y$. If $zw < 0$, there are no solutions, but if $zw > 0$, these simultaneous equations always have exactly two solutions and these vary with whether both $z$ and $w$ are positive or both are negative. If both $z > 0$ and $w > 0$, the solutions are:

$$x = \sqrt{zw},\ y = \sqrt{z/w}, \qquad \text{and} \qquad x = -\sqrt{zw},\ y = -\sqrt{z/w}$$

If both $z < 0$ and $w < 0$, the solutions are:

$$x = \sqrt{zw},\ y = -\sqrt{z/w}, \qquad \text{and} \qquad x = -\sqrt{zw},\ y = \sqrt{z/w}$$

[The case of $zw = 0$ can be ignored as $\mathbf{X}$ and $\mathbf{Y}$ are joint continuous.]

The Jacobian determinant of the transformation is

$$J(x, y) = \det \begin{bmatrix} y & x \\ 1/y & -x/y^2 \end{bmatrix} = -2x/y.$$

At the solution points the absolute value of this is $2|w|$. Hence,

$$f_{\mathbf{ZW}}(z, w) = \begin{cases} \dfrac{f_{\mathbf{XY}}(\sqrt{zw}, \sqrt{z/w}) + f_{\mathbf{XY}}(-\sqrt{zw}, -\sqrt{z/w})}{2|w|}, & \text{if } z > 0 \text{ and } w > 0; \\ \dfrac{f_{\mathbf{XY}}(\sqrt{zw}, -\sqrt{z/w}) + f_{\mathbf{XY}}(-\sqrt{zw}, \sqrt{z/w})}{2|w|}, & \text{if } z < 0 \text{ and } w < 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \dfrac{2}{\pi|w|} e^{-(zw+[z/w])} = \dfrac{2}{\pi|w|} e^{-z(w+[1/w])}, & \text{if } z > 0 \text{ and } w > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$f_{\mathbf{W}}(w) = \int_{-\infty}^{\infty} f_{\mathbf{ZW}}(z, w)\, dz = \begin{cases} \dfrac{2}{\pi|w|} \displaystyle\int_0^{\infty} e^{-z(w+[1/w])}\, dz, = \dfrac{2}{\pi w(w + [1/w])}, & \text{if } w > 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$= \frac{2}{\pi(1 + w^2)} u(w).$$

For $w > 0$, we thus have that

$$f_{\mathbf{Z}}(z \mid \mathbf{W} = w) = \frac{f_{\mathbf{WZ}}(z, w)}{f_{\mathbf{W}}(w)} = \frac{1 + w^2}{|w|} e^{-z(w+[1/w])} u(z),$$

which varies with both $z$ and $w$ [for $w = 1$ this is $2e^{-2z}u(z)$ and for $w = 2$ this is $\frac{5}{2}e^{-5z/2}u(z)$] and so cannot be $f_{\mathbf{Z}}(z)$. Thus $f_{\mathbf{Z}}(z) f_{\mathbf{W}}(w) \neq f_{\mathbf{ZW}}(z, w)$ for some $z, w$, giving us that $\mathbf{Z}$ and $\mathbf{W}$ are *not* independent.

2/ First we solve the simultaneous equations $z = x^2y^2, \quad w = y^2/x^2$. If $zw < 0$, there are no solutions, but if $zw > 0$, these simultaneous equations always have exactly two solutions and these vary with whether both $z$ and $w$ are positive or both negative. If both $z > 0$ and $w > 0$, the solutions are:

$$x = \sqrt[4]{z/w},\ y = \sqrt[4]{zw}; \quad x = \sqrt[4]{z/w},\ y = -\sqrt[4]{zw}; \quad x = -\sqrt[4]{z/w},\ y = \sqrt[4]{zw}; \quad \text{and} \quad x = -\sqrt[4]{z/w},\ y = -\sqrt[4]{zw}.$$

Otherwise there are no solutions unless both $z = 0$ and $w = 0$ in which case there are an uncountable number of solutions and then $x$ can have any value as long as $y = 0$.

The Jacobian determinant of the transformation is

$$J(x, y) = \det \begin{bmatrix} 2xy^2 & 2x^2y \\ -2y^2/x^3 & 2y/x^2 \end{bmatrix} = 8y^3/x.$$

At the solution points, the absolute value of this is $8\sqrt[4]{z^2w^4} = 8w\sqrt{z}$. Hence,

$$f_{\mathbf{ZW}}(z, w) = \begin{cases} \dfrac{f_{\mathbf{XY}}(\sqrt[4]{z/w}, \sqrt[4]{zw}) + f_{\mathbf{XY}}(-\sqrt[4]{z/w}, \sqrt[4]{zw}) + f_{\mathbf{XY}}(\sqrt[4]{z/w}, -\sqrt[4]{zw}) + f_{\mathbf{XY}}(-\sqrt[4]{z/w}, -\sqrt[4]{zw})}{8w\sqrt{z}}, & z, w > 0; \\ 0, & \text{otherwise.} \end{cases}$$

But as this $f_{\mathbf{XY}}(x, y)$ is only nonzero if both $x > 0$ and $y > 0$, this simplifies to

$$f_{\mathbf{ZW}}(z, w) = \begin{cases} \dfrac{f_{\mathbf{XY}}(\sqrt[4]{z/w}, \sqrt[4]{zw})}{8w\sqrt{z}}, & \text{if } z > 0,\ w > 0; \\ 0, & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \dfrac{(\sqrt[4]{zw} + 1)e^{-(\sqrt[4]{z/w} + \sqrt[4]{zw} + \sqrt{z})}}{8w\sqrt{z}}, & \text{if } z > 0,\ w > 0; \\ 0, & \text{otherwise.} \end{cases}$$

We have ignored the case that $z = w = 0$ as the event that $\{\mathbf{X} = 0 \text{ or } \mathbf{Y} = 0\}$ is a zero probability event.

3/ Let us select as an auxiliary random variable, $\mathbf{W} = \mathbf{X}$. Now let us find the joint density function of $\mathbf{Z}$, $\mathbf{W}$. First we solve the simultaneous equations $z = xy, w = x$, which have a unique solution $x = w, y = z/w$ (if $w \neq 0$). The Jacobian determinant of the transformation is

$$J(x, y) = \det \begin{bmatrix} y & x \\ 1 & 0 \end{bmatrix} = -x.$$

At the solution points the absolute value of this is $|w|$. Hence, since

$$f_{\mathbf{XY}}(x, y) = \begin{cases} 1, & \text{if } 0 < x < 1,\ 0 < y < 1; \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$f_{\mathbf{ZW}}(z, w) = f_{\mathbf{XY}}(w, z/w) = \begin{cases} \dfrac{1}{|w|}, & \text{if } 0 < z < w < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$f_{\mathbf{Z}}(z) = \iint_{-\infty}^{\infty} f_{\mathbf{ZW}}(z, w)\, dw = \begin{cases} \displaystyle\int_z^1 \frac{1}{w}\, dw = -\ln z, & \text{if } 0 < z < 1; \\ 0, & \text{otherwise.} \end{cases}$$

4/ (a) $\rho_{\mathbf{XY}} \triangleq \dfrac{\text{cov}(\mathbf{X},\mathbf{Y})}{\sqrt{\text{var}(\mathbf{X})\text{var}(\mathbf{Y})}} = \dfrac{-1}{\sqrt{12}} \simeq -0.288675.$

(b) In general for real random variables $\mathbf{A}$ and $\mathbf{B}$, $\text{cor}(\mathbf{A},\mathbf{B}) = \text{cov}(\mathbf{A},\mathbf{B}) + \mathcal{E}\{\mathbf{A}\}\mathcal{E}\{\mathbf{B}\}$, so

$$\text{cor}(\mathbf{X},\mathbf{X}) = \text{var}(\mathbf{X}) + [\mathcal{E}\{\mathbf{X}\}]^2 = 3 + 2^2 = 7, \qquad \text{cor}(\mathbf{Y},\mathbf{Y}) = \text{var}(\mathbf{Y}) + [\mathcal{E}\{\mathbf{Y}\}]^2 = 4 + (-3)^2 = 13,$$

$$\text{cor}(\mathbf{X},\mathbf{Y}) = \text{cor}(\mathbf{Y},\mathbf{X}) = \text{cov}(\mathbf{X},\mathbf{Y}) + \mathcal{E}\{\mathbf{X}\}\mathcal{E}\{\mathbf{Y}\} = -1 + 2\times(-3) = -7.$$

Thus

$$\mathbf{R}_{\mathbf{XY}} = \begin{bmatrix} \text{cor}(\mathbf{X},\mathbf{X}) & \text{cor}(\mathbf{X},\mathbf{Y}) \\ \text{cor}(\mathbf{Y},\mathbf{X}) & \text{cor}(\mathbf{Y},\mathbf{Y}) \end{bmatrix} = \begin{bmatrix} 7 & -7 \\ -7 & 13 \end{bmatrix}.$$

(c)

$$\mathbf{C}_{\mathbf{XY}} = \begin{bmatrix} \text{cov}(\mathbf{X},\mathbf{X}) & \text{cov}(\mathbf{X},\mathbf{Y}) \\ \text{cov}(\mathbf{Y},\mathbf{X}) & \text{cov}(\mathbf{Y},\mathbf{Y}) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}.$$

(d) $\mathcal{E}\{\mathbf{Z}\} = \mathcal{E}\{\mathbf{X}\} - 2\mathcal{E}\{\mathbf{Y}\} + 1 = 2 + 6 + 1 = 9.$

$\text{var}(\mathbf{Z}) = \text{var}(\mathbf{X}) + (-2)^2\text{var}(\mathbf{Y}) + 2\times(-2)\times 1\text{cov}(\mathbf{X},\mathbf{Y}) = 3 + 16 + 4 = 23.$

mean power $= \mathcal{E}\{\mathbf{Z}^2\} = \text{var}(\mathbf{Z}) + (\mathcal{E}\{\mathbf{Z}\})^2 = 23 + 81 = 104.$

5/ $\cos^2(x/2) = \frac{1}{2} + \frac{1}{2}\cos(x), \qquad \sin^2(x/2) = \frac{1}{2} - \frac{1}{2}\cos(x), \qquad \cos^2(x/2)\sin^2(x/2) = \frac{1}{4}(1 - \cos^2(x)) = \frac{1}{8} - \frac{1}{8}\cos(2x).$

$$\mathcal{E}\{\mathbf{X}\} = \tfrac{1}{2} + \int_0^\pi \frac{\cos(\phi)}{2\pi}\,d\phi = \tfrac{1}{2}, \qquad \mathcal{E}\{\mathbf{Y}\} = \tfrac{1}{2} - \int_0^\pi \frac{\cos(\phi)}{2\pi}\,d\phi = \tfrac{1}{2}.$$

$$\text{cor}(\mathbf{X},\mathbf{Y}) = \int_0^\pi \frac{\sin^2(\phi/2)\cos^2(\phi/2)}{\pi}\,d\phi = \tfrac{1}{8} - \int_0^\pi \frac{\cos(2\phi)}{18\pi}\,d\phi = \tfrac{1}{8}.$$

$$\text{cov}(\mathbf{X},\mathbf{Y}) = \mathcal{E}\{\mathbf{XY}\} - \mathcal{E}\{\mathbf{X}\}\mathcal{E}\{\mathbf{Y}\} = \tfrac{1}{8} - \tfrac{1}{4} = -\tfrac{1}{8} \neq 0.$$

Since $\text{cov}(\mathbf{X},\mathbf{Y}) \neq 0$, the two random variables are not uncorrelated.

6/ (a) The value of $\alpha$ that minimizes $\mathcal{E}\{|\mathbf{X} + \alpha\mathbf{Y}]^2\}$ much satisfy the condition for a local maximum which is that $\dfrac{\partial}{\partial\alpha}\mathcal{E}\{[\mathbf{X} + \alpha\mathbf{Y}]^2\} = 0$. As $\mathcal{E}\{[\mathbf{X} + \alpha\mathbf{Y}]^2\} = \mathcal{E}\{\mathbf{X}^2\} + \alpha^2\mathcal{E}\{\mathbf{Y}^2\} + 2\alpha\mathcal{E}\{\mathbf{XY}\}$, we have

$$\frac{\partial}{\partial x}\mathcal{E}\{[\mathbf{X} + \alpha\mathbf{Y}]^2\} = 2\alpha\mathcal{E}\{\mathbf{Y}^2\} + 2\mathcal{E}\{\mathbf{XY}\} = 0. \qquad (*)$$

By assumption $\mathcal{E}\{\mathbf{Y}^2\} > 0$, so $(*)$ has a unique solution:

$$\alpha = \frac{-\mathcal{E}\{\mathbf{XY}\}}{\mathcal{E}\{\mathbf{Y}^2\}} = \frac{-\text{cor}(\mathbf{X},\mathbf{Y})}{\mathcal{E}\{\mathbf{Y}^2\}}. \qquad (**)$$

(b) At the minimum, the value of $\mathcal{E}\{[\mathbf{X} + \alpha\mathbf{Y}]^2\}$ is

$$\mathcal{E}\{\mathbf{X}^2\} + \frac{\mathcal{E}^2\{\mathbf{XY}\}}{\mathcal{E}\{\mathbf{Y}^2\}} - \frac{2\mathcal{E}^2\{\mathbf{XY}\}}{\mathcal{E}\{\mathbf{Y}^2\}} = \mathcal{E}\{\mathbf{X}^2\} - \frac{\mathcal{E}^2\{\mathbf{XY}\}}{\mathcal{E}\{\mathbf{Y}^2\}}.$$

Since $\mathcal{E}\{[\mathbf{X} + \alpha\mathbf{Y}]^2\} \geq 0$ for any $\alpha$, we must conclude $\mathcal{E}\{\mathbf{X}^2\} \geq \dfrac{\mathcal{E}^2\{\mathbf{XY}\}}{\mathcal{E}\{\mathbf{Y}^2\}}$. Since $\mathcal{E}\{\mathbf{Y}^2\} > 0$, this gives us that

$$\mathcal{E}^2\{\mathbf{XY}\} \leq \mathcal{E}\{\mathbf{X}^2\}\mathcal{E}\{\mathbf{Y}^2\} \qquad \text{if } \mathcal{E}\{\mathbf{Y}^2\} > 0.$$

Now if we interchange $\mathbf{X}$ and $\mathbf{Y}$, we would get

$$\mathcal{E}^2\{\mathbf{XY}\} \leq \mathcal{E}\{\mathbf{X}^2\}\mathcal{E}\{\mathbf{Y}^2\} \qquad \text{if } \mathcal{E}\{\mathbf{X}^2\} > 0,$$

6/ (b) continued)

establishing that (†) holds when at least one of $\mathcal{E}\{\mathbf{X}^2\} > 0$ or $\mathcal{E}\{\mathbf{Y}^2\} > 0$ holds. We can also show it holds in the final case where $\mathcal{E}\{\mathbf{X}^2\} = 0$ and $\mathcal{E}\{\mathbf{Y}^2\} = 0$. If $\mathcal{E}\{\mathbf{X}^2\} = \mathcal{E}\{\mathbf{Y}^2\} = 0$, then

$$\mathcal{E}\{[\mathbf{X}+\mathbf{Y}]^2\} = \mathcal{E}\{\mathbf{X}^2\} + \mathcal{E}\{\mathbf{Y}^2\} + 2\mathcal{E}\{\mathbf{XY}\} = 2\mathcal{E}\{\mathbf{XY}\} \geq 0 \qquad \Longrightarrow \mathcal{E}\{\mathbf{XY}\} \geq 0; \qquad (A)$$
$$\mathcal{E}\{[\mathbf{X}-\mathbf{Y}]^2\} = \mathcal{E}\{\mathbf{X}^2\} + \mathcal{E}\{\mathbf{Y}^2\} - 2\mathcal{E}\{\mathbf{XY}\} = -2\mathcal{E}\{\mathbf{XY}\} \geq 0 \qquad \Longrightarrow \mathcal{E}\{\mathbf{XY}\} \leq 0. \qquad (B)$$

For $(A)$ and $(B)$ to hold, we must have $\mathcal{E}\{\mathbf{XY}\} = 0$, allowing us to state

$$\mathcal{E}^2\{\mathbf{XY}\} \leq \mathcal{E}\{\mathbf{X}^2\}\mathcal{E}\{\mathbf{Y}^2\} \qquad \text{if } \mathcal{E}\{\mathbf{X}^2\} = \mathcal{E}\{\mathbf{Y}^2\} = 0$$

So we know that (†) holds in all cases where $\mathbf{X}$ and $\mathbf{Y}$ that have finite second order moments.

To have equality hold in the inequality when one of $\mathcal{E}\{\mathbf{X}^2\} > 0$ or $\mathcal{E}\{\mathbf{Y}^2\} > 0$ holds, we see we must have for some $\alpha$ either $\mathcal{E}\{[\mathbf{X}+\alpha\mathbf{Y}]^2\} = 0 \implies \mathbf{X} = -\alpha\mathbf{Y}$ or $\mathcal{E}\{[\mathbf{Y}+\alpha\mathbf{X}]^2\} = 0 \implies \mathbf{Y} = -\alpha\mathbf{X}$ that is to say, one random variable is a scaled version of the other. Thinking about a random variable as a vector, this can be stated as the random variables are colinear.

(c) If we replace $\mathbf{X}$ by $\mathbf{X} - \mathcal{E}\{\mathbf{X}\}$ and $\mathbf{Y}$ by $\mathbf{Y} - \mathcal{E}\{\mathbf{Y}\}$ in the above result gives us the required result.

7/ $\mathcal{E}\{\mathbf{Z}\} = \mathcal{E}\{\mathbf{X}\} + 2\mathcal{E}\{\mathbf{Y}\} - 3 = 1 - 2 - 3 = -4$, $\mathcal{E}\{\mathbf{W}\} = 3\mathcal{E}\{\mathbf{X}\} - 7\mathcal{E}\{\mathbf{Y}\} + 12 = 3 + 7 + 12 = 22$.

$$\mathbf{C_{ZW}} = \begin{bmatrix} 1 & 2 \\ 3 & -7 \end{bmatrix} \mathbf{C_{XY}} \begin{bmatrix} 1 & 3 \\ 2 & -7 \end{bmatrix} = \begin{bmatrix} 34 & -119 \\ -119 & 501 \end{bmatrix}$$

$$\mathbf{R_{ZW}} = \mathbf{C_{ZW}} + \begin{bmatrix} -4 \\ 22 \end{bmatrix} \begin{bmatrix} -4 & 22 \end{bmatrix} = \begin{bmatrix} 50 & -207 \\ -207 & 985 \end{bmatrix}$$