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École de science informatique
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School of Electrical Engineering
and Computer Science

ELG 3126 **RANDOM SIGNALS AND SYSTEMS** **Winter 2023**

ASSIGNMENT 10

(due at 8:30 AM Thursday, March 30 in class)

1. Suppose that $\mathbf{X}$, $\mathbf{Y}$ are two random variables with means 1 and 2 (respectively) and correlation matrix

$$\mathbf{R_{XY}} = \begin{bmatrix} 16 & 3 \\ 3 & 9 \end{bmatrix}$$

Find the covariance matrix $\mathbf{C_{XY}}$. Also, find the mean, mean square value, and variance of $\mathbf{W} = 3\mathbf{X} - 2\mathbf{Y} - 1$.

2. A complex-valued random variable $\mathbf{Z}$ is defined by $\mathbf{Z} \triangleq \mathbf{X} + j\mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ are independent identically distributed real random variables uniformly distributed on $(0, 2)$.
   (a) Find the mean and variance of $\mathbf{Z}$.
   (b) What is the covariance of $\mathbf{Z}$ and $\mathbf{W} \triangleq \mathbf{X} - j\mathbf{Y}$. Are $\mathbf{Z}$ and $\mathbf{W}$ uncorrelated? orthogonal?

3. Suppose $\boldsymbol{\theta} \sim U(-\pi, \pi,)$, and $\mathbf{Z}$ and $\mathbf{W}$ are two complex random variables defined by $\mathbf{Z} \triangleq \exp(j\boldsymbol{\theta})$ and $\mathbf{W} \triangleq \exp(-j\boldsymbol{\theta})$. Find (a) the variance of $\mathbf{Z}$ and (b) of $\mathbf{W}$, (c) their correlation, (d) covariance and (e) their correlation coefficient. Repeat this for the case that $\boldsymbol{\theta} \sim U(-\frac{1}{2}\pi, \frac{1}{2}\pi,)$.

4. Suppose that $\mathbf{X}$, $\mathbf{Y}$ are two jointly Gaussian random variables with means 2 and 1 (respectively) and covariance matrix

$$\mathbf{C_{XY}} = \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}$$

Find the mean, mean square value, and variance of $\mathbf{W} = 2\mathbf{X} - 3\mathbf{Y} + 2$. Find the density function of $\mathbf{W}$ and the numeric value of $\mathcal{P}(\mathbf{W} > 0)$.

5. Suppose that $\mathbf{X}$, $\mathbf{Y}$ are two jointly Gaussian random variables with means 0 and 1 (respectively) and covariance matrix

$$\mathbf{C_{XY}} = \begin{bmatrix} 4 & 1 \\ 1 & 16 \end{bmatrix}$$

Find the density function of $\mathbf{Z} = \mathbf{X}^2 + 4\mathbf{Y}^2 + 4\mathbf{XY} + 2\mathbf{X} + 4\mathbf{Y}$.
Hint: Observe that $\mathbf{Z} = (\mathbf{X} + 2\mathbf{Y} + 1)^2 - 1$ to make this simple by first considering $\mathbf{W} = \mathbf{X} + 2\mathbf{Y} + 1$ and its distribution..

6. Suppose $\mathbf{X}$ and $\mathbf{Y}$ are two jointly Guassian random variable with joint pdf

$$f_{\mathbf{XY}}(x, y) = \alpha e^{-(4x^2 + y^2 + 2xy - 4x - 2y + 4)/4}.$$

Find the value of α, $\mathcal{E}\{\mathbf{X}\}$, $\mathcal{E}\{\mathbf{Y}\}$, $\mathrm{var}(\mathbf{X})$, $\mathrm{var}(\mathbf{Y})$ and $\mathrm{cov}(\mathbf{X}, \mathbf{Y})$.
Hint: Start by finding the means of $\mathbf{X}$, $\mathbf{Y}$ noting that the joint density function always peaks at $(\mu_{\mathbf{X}}, \mu_{\mathbf{Y}})$. Then compare the form of the density function to the general form of the density function of jointly Gaussian random variables to determine all the parameters.

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ELG 3126 **RANDOM SIGNALS AND SYSTEMS** **Winter 2022**

ASSIGNMENT 10
SOLUTIONS

1/ $\mathcal{E}\{[\mathbf{X} \quad \mathbf{Y}]\} = [1 \quad 2]$. Since $\mathbf{C_{XY}} = \mathbf{R_{XY}} - \mathcal{E}\{[\mathbf{X},\mathbf{Y}]\}^{\mathrm{T}}\mathcal{E}\{[\mathbf{X},\mathbf{Y}]\}$ we have

$$\mathbf{C_{XY}} = \begin{bmatrix} 16 & 3 \\ 3 & 9 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} [1 \quad 2] = \begin{bmatrix} 16 & 3 \\ 3 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 15 & 1 \\ 1 & 5 \end{bmatrix};$$

$$\mathcal{E}\{\mathbf{W}\} = 3\mathcal{E}\{\mathbf{X}\} - 2\mathcal{E}\{\mathbf{Y}\} - 1 = 3 - 4 - 1 = -2;$$

$$\mathrm{var}(\mathbf{W}) = \mathrm{var}(3\mathbf{X} - 2\mathbf{Y}) = [3 \quad -2]\,\mathbf{C_{XY}} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = [3 \quad -2] \begin{bmatrix} 43 \\ -7 \end{bmatrix} = 143;$$

$$\mathcal{E}\{\mathbf{W}^2\} = \mathrm{var}(\mathbf{W}) + \mathcal{E}^2\{\mathbf{W}\} = 143 + 4 = 147.$$

2/ (a) $\mathcal{E}\{\mathbf{X}\} = \mathcal{E}\{\mathbf{Y}\} = 1$, so $\mathcal{E}\{\mathbf{Z}\} \triangleq \mathcal{E}\{\mathbf{X}\} + j\mathcal{E}\{\mathbf{Y}\} = 1 + j$. Thus $\mathrm{var}(\mathbf{Z}) = \mathcal{E}\{|\mathbf{Z}|^2\} - |\mathcal{E}\{\mathbf{Z}\}|^2 = \mathcal{E}\{|\mathbf{Z}|^2\} - 2$. $\mathcal{E}\{|\mathbf{Z}|^2\} = \mathcal{E}\{\mathbf{X}^2 + \mathbf{Y}^2\} = \mathcal{E}\{\mathbf{X}^2\} + \mathcal{E}\{\mathbf{Y}^2\} = 2\mathcal{E}\{\mathbf{X}^2\}$ since $\mathbf{X}$ and $\mathbf{Y}$ have the same distribution.

$$\mathcal{E}\{\mathbf{X}^2\} = \int_0^2 \tfrac{1}{2}x^2 = \tfrac{1}{6}x^3\Big|_0^2 = \tfrac{4}{3}.$$

Hence $\mathrm{var}(\mathbf{Z}) = \frac{2}{3}$.

(b) $\mathrm{cov}(\mathbf{Z},\mathbf{W}) = \mathcal{E}\{\mathbf{Z}^*\mathbf{W}\} - \mathcal{E}\{\mathbf{Z}^*\}\mathcal{E}\{\mathbf{W}\} = \mathcal{E}\{(\mathbf{X} - j\mathbf{Y})^2\} - \mathcal{E}\{\mathbf{X} - j\mathbf{Y}\}^2 = \mathcal{E}\{\mathbf{X}^2 - \mathbf{Y}^2\} - 2j\mathcal{E}\{\mathbf{XY}\} - (1-j)^2$, Thus, since $\mathbf{X}$ and $\mathbf{Y}$ are independent identically distributed random variables,

$$\mathrm{cov}(\mathbf{Z},\mathbf{W}) = -2j\mathcal{E}\{\mathbf{X}\}\mathcal{E}\{\mathbf{Y}\} + 2j = -2j + 2j = 0.$$
$$\mathrm{cor}(\mathbf{Z},\mathbf{W}) = \mathrm{cov}(\mathbf{Z},\mathbf{W}) + (1-j)^2 = -2j.$$

The random variables $\mathbf{Z}$ and $\mathbf{W}$ are uncorrelated but not orthogonal.

3/ Since $\mathbf{Z}$ is a function of $\boldsymbol{\theta}$, the mean of $\mathbf{Z}$ the mean can be calculated in terms of the density function of $\boldsymbol{\theta}$:

$$\mathcal{E}\{\mathbf{Z}\} = \int_{-\infty}^{\infty} e^{j\phi} f_{\boldsymbol{\theta}}(\phi)\,d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\phi}\,d\phi = \frac{1}{j2\pi}e^{j\phi}\Big|_{-\pi}^{\pi} = 0.$$

Likewise

$$\mathcal{E}\{\mathbf{W}\} = \int_{-\infty}^{\infty} e^{-j\phi} f_{\boldsymbol{\theta}}(\phi)\,d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\phi}\,d\phi = \frac{-1}{j2\pi}e^{-j\phi}\Big|_{-\pi}^{\pi} = 0;$$

$$\mathrm{cor}(\mathbf{Z},\mathbf{Z}) = \mathcal{E}\{|\mathbf{Z}|^2\} = \int_{-\infty}^{\infty} |e^{j\phi}|^2 f_{\boldsymbol{\theta}}(\phi)\,d\phi = \int_{-\infty}^{\infty} f_{\boldsymbol{\theta}}(\phi)\,d\phi = 1;$$

$$\mathrm{cor}(\mathbf{W},\mathbf{W}) = \mathcal{E}\{|\mathbf{W}|^2\} = \int_{-\infty}^{\infty} |e^{-j\phi}|^2 f_{\boldsymbol{\theta}}(\phi)\,d\phi = \int_{-\infty}^{\infty} f_{\boldsymbol{\theta}}(\phi)\,d\phi = 1;$$

$$\mathrm{cor}(\mathbf{Z},\mathbf{W}) = \mathcal{E}\{\mathbf{Z}^*\mathbf{W}\} = \int_{-\infty}^{\infty} [e^{-j\phi}]^2 f_{\boldsymbol{\theta}}(\phi)\,d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j2\phi}\,d\phi = 0.$$

. . . *(problem continued on the next page)*

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3/ (continued)
This then give us that

$$\begin{aligned}
\mathrm{var}(\mathbf{Z}) &= \mathrm{cor}(\mathbf{Z},\mathbf{Z}) - |\mathcal{E}\{\mathbf{Z}\}|^2 = 1,\\
\mathrm{var}(\mathbf{W}) &= \mathrm{cor}(\mathbf{W},\mathbf{W}) - |\mathcal{E}\{\mathbf{W}\}|^2 = 1,\\
\mathrm{cov}(\mathbf{Z},\mathbf{W}) &= \mathrm{cor}(\mathbf{Z},\mathbf{W}) - [\mathcal{E}\{\mathbf{Z}\}]^*\mathcal{E}\{\mathbf{W}\} = 0, \quad \Longrightarrow \quad \rho_{\mathbf{ZW}} = \frac{\mathrm{cov}(\mathbf{Z},\mathbf{W})}{\sqrt{\mathrm{var}(\mathbf{Z})\mathrm{var}(\mathbf{W})}} = 0.
\end{aligned}$$

Now if we repeat this for $\boldsymbol{\theta} \sim U(-\pi/2, \pi/2)$ these results become

$$\begin{aligned}
\mathcal{E}\{\mathbf{Z}\} &= \int_{-\infty}^{\infty} e^{j\phi} f_{\boldsymbol{\theta}}(\phi)\,d\phi = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{j\phi}\,d\phi = \frac{1}{j\pi}e^{j\phi}\Big|_{-\pi/2}^{\pi/2} = \frac{2}{\pi} \simeq 0.63662;\\
\mathcal{E}\{\mathbf{W}\} &= \int_{-\infty}^{\infty} e^{-j\phi} f_{\boldsymbol{\theta}}(\phi)\,d\phi = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{-j\phi}\,d\phi = \frac{-1}{j\pi}e^{-j\phi}\Big|_{-\pi/2}^{\pi/2} = \frac{2}{\pi} \simeq 0.63662;\\
\mathrm{cor}(\mathbf{Z},\mathbf{Z}) &= \mathcal{E}\{|\mathbf{Z}|^2\} = \int_{-\infty}^{\infty} |e^{j\phi}|^2 f_{\boldsymbol{\theta}}(\phi)\,d\phi = \int_{-\infty}^{\infty} f_{\boldsymbol{\theta}}(\phi)\,d\phi = 1;\\
\mathrm{cor}(\mathbf{W},\mathbf{W}) &= \mathcal{E}\{|\mathbf{W}|^2\} = \int_{-\infty}^{\infty} |e^{-j\phi}|^2 f_{\boldsymbol{\theta}}(\phi)\,d\phi = \int_{-\infty}^{\infty} f_{\boldsymbol{\theta}}(\phi)\,d\phi = 1;\\
\mathrm{cor}(\mathbf{Z},\mathbf{W}) &= \mathcal{E}\{\mathbf{Z}^*\mathbf{W}\} = \int_{-\infty}^{\infty} [e^{-j\phi}]^2 f_{\boldsymbol{\theta}}(\phi)\,d\phi = \frac{-1}{j2\pi}e^{-j2\phi}\Big|_{-\pi/2}^{\pi/2} = 0;\\
\mathrm{var}(\mathbf{Z}) &= \mathrm{cor}(\mathbf{Z},\mathbf{Z}) - |\mathcal{E}\{\mathbf{Z}\}|^2 = 1 - \left(\frac{2}{\pi}\right)^2 \simeq 0.59472;\\
\mathrm{var}(\mathbf{W}) &= \mathrm{cor}(\mathbf{W},\mathbf{W}) - |\mathcal{E}\{\mathbf{W}\}|^2 = 1 - \left(\frac{2}{\pi}\right)^2 \simeq 0.59472;\\
\mathrm{cov}(\mathbf{Z},\mathbf{W}) &= \mathrm{cor}(\mathbf{Z},\mathbf{W}) - [\mathcal{E}\{\mathbf{Z}\}]^*\mathcal{E}\{\mathbf{W}\} = 0 - \left(\frac{2}{\pi}\right)^2 \simeq -0.40528;\\
\rho_{\mathbf{ZW}} &= \frac{\mathrm{cov}(\mathbf{Z},\mathbf{W})}{\sqrt{\mathrm{var}(\mathbf{Z})\mathrm{var}(\mathbf{W})}} = \frac{-4}{4-\pi^2} \simeq -0.68148.
\end{aligned}$$

4/ We know that since **X** and **Y** are jointly Gaussian that **W** is a Gaussian random variable. The mean of **W** is

$$\mathcal{E}\{\mathbf{W}\} = 2\mathcal{E}\{\mathbf{X}\} - 3\mathcal{E}\{\mathbf{Y}\} + 2 = 3.$$

The variance of **W** is the variance of $2\mathbf{X} - 3\mathbf{Y}$ which is

$$\mathrm{var}(2\mathbf{X} - 3\mathbf{Y}) = 2^2\mathrm{var}(\mathbf{X}) + (-3)^2\mathrm{var}(\mathbf{Y}) + 2 \times 2 \times (-3)\,\mathrm{cov}(\mathbf{X},\mathbf{Y}) = 16 + 81 + 12 = 109.$$

Thus the mean square value of **W** is $\mathcal{E}\{\mathbf{W}^2\} = \mathrm{var}(\mathbf{W}) + \mathcal{E}^2\{\mathbf{W}\} = 118$. The density function of **W** is then that of a N(3,109) random variable:

$$f_{\mathbf{W}}(w) = \frac{1}{\sqrt{218\pi}}e^{-(w-3)^2/218}.$$

and so $\mathcal{P}(\mathbf{W} > 0) = Q([0-3]/\sqrt{109}) = 1 - Q(3/\sqrt{109}) \simeq 1 - Q(0.2873) \simeq 0.6131$.

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5/ Define $\mathbf{W} = \mathbf{X} + 2\mathbf{Y} + 1$ so that $\mathbf{Z} = (\mathbf{X} + 2\mathbf{Y} + 1)^2 - 1 = \mathbf{W}^2 - 1$. We can then find the density function of $\mathbf{Z}$ by first finding the density of $\mathbf{W}$, the the density function of $\mathbf{W}^2$ and finally of the density function of $\mathbf{Z}$:

$$f_{\mathbf{Z}}(z) = f_{\mathbf{W}^2-1}(z) = f_{\mathbf{W}^2}(z+1), \qquad f_{\mathbf{W}^2}(w) = \begin{cases} \frac{f_{\mathbf{W}}(\sqrt{w}) + f_{\mathbf{W}}(-\sqrt{w})}{2\sqrt{w}}, & \text{if } w > 0 \\ 0. & \text{otherwise.} \end{cases}$$

We know that since $\mathbf{X}$ and $\mathbf{Y}$ are jointly Gaussian, $\mathbf{W}$ is a Gaussian random variable. The mean of $\mathbf{W}$ is

$$\mathcal{E}\{\mathbf{W}\} = \mathcal{E}\{\mathbf{X}\} + 2\mathcal{E}\{\mathbf{Y}\} + 1 = 3,$$

and the variance of $\mathbf{W}$ is the variance of $\mathbf{X} + 2\mathbf{Y}$:

$$\text{var}(\mathbf{X} + 2\mathbf{Y}) = \text{var}(\mathbf{X}) + 2^2\text{var}(\mathbf{Y}) + 2 \times 2\,\text{cov}(\mathbf{X}, \mathbf{Y}) = 4 + 64 + 4 = 72.$$

Thus the density function of $\mathbf{W}$ is then that of a N(3,72) random variable:

$$f_{\mathbf{W}}(w) = \frac{1}{\sqrt{144\pi}} e^{-(w-3)^2/144}.$$

Thus the density function of $\mathbf{W}^2$ is

$$f_{\mathbf{W}^2}(w) = \begin{cases} \frac{e^{-(\sqrt{w}-3)^2/144} + e^{-(\sqrt{w}+3)^2/144}}{24\sqrt{\pi w)}}, & \text{if } w > 0; \\ 0, & \text{otherwise.} \end{cases}$$

We then finally have

$$f_{\mathbf{Z}}(z) = \begin{cases} \frac{e^{-(\sqrt{z+1}-3)^2/144} + e^{-(\sqrt{z+1}+3)^2/144}}{24\sqrt{\pi(z+1)}}, & \text{if } z > -1; \\ 0, & \text{otherwise;} \end{cases}$$

which simplifies to

$$f_{\mathbf{z}}(z) = \begin{cases} \frac{e^{-(z+10)/144}}{24\sqrt{\pi(z+1)}}[e^{-\sqrt{z+1}/24} + e^{\sqrt{z+1}/24}], & \text{if } z > -1; \\ 0, & \text{otherwise.} \end{cases}$$

6/ The simplest way to find the required information is to first note that the joint density function of jointly Gaussian random variables $f_{\mathbf{XY}}(x, y)$ peaks at $x = \mathcal{E}\{\mathbf{X}\},\ y = \mathcal{E}\{\mathbf{Y}\}$. The given function peaks where $(4x^2 + y^2 + 2xy - 4x - 2y + 4)$ has its minimum. We find this by setting the gradient of the function to zero:

$$\frac{\partial}{\partial x}(4x^2 + y^2 + 2xy - 4x - 2y + 4) = 8x + 2y - 4 = 0$$
$$\frac{\partial}{\partial y}(4x^2 + y^2 + 2xy - 4x - 2y + 4) = 2y + 2x - 2 = 0.$$

The solution to these simultaneous equations is $x = \frac{1}{3}$, $y = \frac{2}{3}$, telling us that $\mathcal{E}\{\mathbf{X}\} = \frac{1}{3},\ \mathcal{E}\{\mathbf{Y}\} = \frac{2}{3}$. Now we will try to express the given function in the standard form for the joint pdf of jointly Gaussian random variables:

$$f_{\mathbf{XY}}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2_{\mathbf{XY}}}\,\sigma_{\mathbf{X}}\sigma_{\mathbf{Y}}} \exp(-\tfrac{1}{2}[\frac{(x-x_0)^2}{\sigma^2_{\mathbf{X}}} + \frac{(y-y_0)^2}{\sigma^2_{\mathbf{Y}}} - \frac{2\rho_{\mathbf{XY}}(x-x_0)(y-y_0)}{\sigma_{\mathbf{X}}\sigma_{\mathbf{Y}}}]/(1-\rho^2_{\mathbf{XY}}))$$

For this $x_0 = \frac{1}{3}$ and $y_0 = \frac{2}{3}$, and we observe

$$(4x^2 + y^2 + 2xy - 4x - 2y + 4) = 4(x - \tfrac{1}{3})^2 + (y - \tfrac{2}{3})^2 + 2(x - \tfrac{1}{3})(y - \tfrac{2}{3}) + \tfrac{8}{3}.$$

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6/ (continued)

Thus

$$\alpha e^{-(4x^2+y^2+2xy-4x-2y+4)/4} = \alpha e^{-2/3} e^{-[(x-\frac{1}{3})^2+\frac{1}{4}(y-\frac{2}{3})^2+\frac{1}{2}(x-\frac{1}{3})(y-\frac{2}{3})]}.$$

To match the form then of the joint density function,

$$2\sigma_{\mathbf{X}}^2(1-\rho_{\mathbf{XY}}^2) = 1, \qquad 2\sigma_{\mathbf{Y}}^2(1-\rho_{\mathbf{XY}}^2) = 4, \qquad \frac{\rho_{\mathbf{XY}}}{(1-\rho_{\mathbf{XY}}^2)\sigma_{\mathbf{X}}\sigma_{\mathbf{Y}}} = \tfrac{1}{2}.$$

From the first two we get $\sigma_{\mathbf{X}}^2\sigma_{\mathbf{Y}}^2(1-\rho_{\mathbf{XY}}^2)^2 = 1 \implies \sigma_{\mathbf{X}}\sigma_{\mathbf{Y}}(1-\rho_{\mathbf{XY}}^2) = 1$, making the last equation $\rho_{\mathbf{XY}} = \frac{1}{2}$. Then $\sigma_{\mathbf{X}}^2 = \mathrm{var}(\mathbf{X}) = \frac{2}{3}$, $\sigma_{\mathbf{Y}}^2 = \mathrm{var}(\mathbf{Y}) = \frac{8}{3}$, and $\mathrm{cov}(\mathbf{X},\mathbf{Y}) = \rho_{\mathbf{XY}}\sigma_{\mathbf{X}}\sigma_{\mathbf{Y}} = \frac{2}{3}$. For the final parameter, α, we note that $\alpha e^{-2/3} = \dfrac{1}{2\pi\sqrt{1-\rho_{\mathbf{XY}}^2}\,\sigma_{\mathbf{X}}\sigma_{\mathbf{Y}}} = \dfrac{\sqrt{3}}{4\pi}$, so $\alpha = \dfrac{e^{2/3}\sqrt{3}}{4\pi}$.

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