



ASSIGNMENT 11

(due at 8:30 AM Thursday, April 6 in class)

1. Three random variables  $\mathbf{X}$ ,  $\mathbf{Y}$  and  $\mathbf{Z}$  have means 0,  $-1$  and 1. respectively. The correlation matrix of the three random variable is

$$\mathbf{R}_{\mathbf{XYZ}} = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 9 & 1 \\ -2 & 1 & 16 \end{bmatrix}$$

- (a) Find the covariance matrix  $\mathbf{C}_{\mathbf{XYZ}}$ .  
(b) Find the mean and variance of  $\mathbf{W} = 2\mathbf{X} - \mathbf{Y} + 3\mathbf{Z} + 3$ . If  $\mathbf{X}$ ,  $\mathbf{Y}$  and  $\mathbf{Z}$  are jointly Gaussian, what is the density function of  $\mathbf{W}$ ?
2. Let  $\tilde{\mathbf{X}} = [\mathbf{X}_1 \ \mathbf{X}_2 \ \mathbf{X}_3 \ \mathbf{X}_4]$ , where the  $\mathbf{X}_i$  are all equal-mean unit-variance independent random variables. Let  $\tilde{\mathbf{Y}} = \mathbf{A}\tilde{\mathbf{X}}$  where

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{6} & \frac{1}{12} \\ 0 & 1 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find the mean vector and covariance matrix of  $\tilde{\mathbf{Y}}$ , and the cross-covariance matrix of  $\tilde{\mathbf{X}}$  and  $\tilde{\mathbf{Y}}$ .

3. The random variables  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$  are independent identically distributed random variable, each with an exponential distribution with parameter  $\alpha > 0$ . Find the density function of  $\mathbf{W} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \dots + \mathbf{X}_n$  for any  $n$ .  
*Hint:* Consider convolution to determine the density function of  $\mathbf{X}_1 + \mathbf{X}_2$ , then  $\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3$ , and generalize.
4. Let  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$  be random variables that have the same finite mean  $\mu$  and whose covariances are given by

$$\text{cov}(\mathbf{X}_i, \mathbf{X}_j) = \begin{cases} \sigma^2, & \text{if } i = j; \\ \rho\sigma^2, & \text{if } |i - j| = 1; \\ 0, & \text{otherwise,} \end{cases}$$

where  $|\rho| < 1$ . Find the mean and variance of  $\mathbf{S}_n = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \dots + \mathbf{X}_n$ . Repeat the calculation for the case where  $\text{cov}(\mathbf{X}_i, \mathbf{X}_j) = \rho^{|i-j|}\sigma^2$ .

5. Suppose that  $\mathbf{X}$  and  $\mathbf{Y}$  are two independent Cauchy random variables:

$$f_{\mathbf{X}}(x) = \frac{\alpha}{\pi} \frac{1}{\alpha^2 + x^2}, \quad f_{\mathbf{Y}}(x) = \frac{\beta}{\pi} \frac{1}{\beta^2 + x^2}, \quad \text{for some } \alpha > 0, \beta > 0.$$

- (a) Find the characteristic function of  $\mathbf{X}$ , and from this the joint characteristic function of  $\mathbf{X}$  and  $\mathbf{Y}$ .  
(b) Find the probability density function of  $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$  using the joint characteristic function found in (a).

. . . (problem continued over)

5. [continued]
- (c) Suppose  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$  are independent identically distributed Cauchy random variables. Find the probability density function of  $\mathbf{Z} = (\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \dots + \mathbf{X}_n)/n$ . Is this approximately Gaussian when  $n$  is large?
6. Let  $\mathbf{S}_n = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \dots + \mathbf{X}_k$ , where  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$  are independent identically distributed chi-squared random variables with  $n_i$  degrees of freedom. Show that  $\mathbf{S}_n$  has a chi-squared distribution with  $n = n_1 + n_2 + n_3 + \dots + n_k$  degrees of freedom.  
*Hint:* Recall the example in class where the chi-squared distribution arose (see problem 7.6(a) in the text), This observation should make this result very easy to prove starting with the result of problem 7.6(a)
7. A student uses a pen for a certain time before he/she loses it or it runs out of ink. It is found that the length of time the student uses a particular pen is governed by an 2-Erlang (gamma) distribution with density function  $f_{\mathbf{X}}(x) = \lambda^2 x e^{-\lambda x} u(x)$ , with parameter value  $\lambda$  so that the mean value is 9 days. The student likes to buy enough pens in August during the back-to-school sales so that he/she has enough pens to last the academic year which is 200 days. How many pens should the student buy in August so that the probability the student will not have to buy more pens during the academic year is at least 0.95? Would your answer change if the distribution of the time a student has the pen was a different distribution (say, a  $\chi^2$  distribution) with a mean of 9 days?  
*Hint:* If we assume that the number of pens will be large, argue that the length of time  $k$  pens will last should be well described by a Gaussian random variable. You may use the results in Table 4.1 in the text in your answer.
8. The sum of 128 randomly chosen real numbers is to be computed. Suppose that each number is rounded-off to the nearest integer, so that the difference between the number and the rounded-off value is well-described as having a uniform distribution on  $[-\frac{1}{2}, \frac{1}{2}]$ . Use the Central Limit Theorem to estimate the probability that the error that would occur if we only added the rounded-off values (instead of adding the actual values) would exceed 4.5 in magnitude.



ASSIGNMENT 11

SOLUTIONS

1/ (a) The mean vector is  $\boldsymbol{\mu}_{\mathbf{XYZ}} = [0 \ 1 \ -1]$ .

$$\begin{aligned} \mathbf{C}_{\mathbf{XYZ}} &= \mathbf{R}_{\mathbf{XYZ}} - \boldsymbol{\mu}_{\mathbf{XYZ}}^T \boldsymbol{\mu}_{\mathbf{XYZ}} = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 9 & 1 \\ -2 & 1 & 16 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} [0 \ 1 \ -1] \\ &= \begin{bmatrix} 4-0 & -1-0 & -2-0 \\ -1-0 & 9-1 & 1+1 \\ -2-0 & 1+1 & 16-1 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 8 & 2 \\ -2 & 2 & 15 \end{bmatrix} \end{aligned}$$

(b)  $\mathcal{E}\{\mathbf{W}\} = 2\mathcal{E}\{\mathbf{X}\} - \mathcal{E}\{\mathbf{Y}\} + 3\mathcal{E}\{\mathbf{Z}\} + 1 = 0 - 1 - 3 + 1 = -3$ .

$$\text{var}(\mathbf{W}) = \text{var}(2\mathbf{X} - \mathbf{Y} + 3\mathbf{Z}) = [2 \ -1 \ 3] \mathbf{C}_{\mathbf{XYZ}} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = [2 \ -1 \ 3] \begin{bmatrix} 3 \\ -4 \\ 39 \end{bmatrix} = 127.$$

Thus  $\mathbf{W} \sim N(-3, 127)$ :  $f_{\mathbf{W}}(w) = \frac{1}{\sqrt{254\pi}} e^{-(w+3)^2/254}$ .

2/ Let each random variable's mean be denoted  $\mu$ .

$$\mathcal{E}\{\tilde{\mathbf{Y}}^T\} = \mathbf{A} \mathcal{E}\{\tilde{\mathbf{X}}^T\} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mu \\ \mu \\ \mu \\ \mu \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 12 \\ 8 \end{bmatrix} \frac{\mu}{8}.$$

$\mathbf{C}_{\tilde{\mathbf{Y}}} = \mathbf{A} \mathbf{C}_{\tilde{\mathbf{X}}} \mathbf{A}^T$ ; from the given information  $\mathbf{C}_{\tilde{\mathbf{X}}} = \mathbf{I}$ , so

$$\mathbf{C}_{\tilde{\mathbf{Y}}} = \mathbf{A} \mathbf{A}^T = \frac{1}{64} \begin{bmatrix} 85 & 42 & 20 & 8 \\ 42 & 84 & 40 & 16 \\ 20 & 40 & 80 & 32 \\ 8 & 16 & 32 & 644 \end{bmatrix}.$$

$$\mathbf{C}_{\tilde{\mathbf{X}}\tilde{\mathbf{Y}}} = \mathbf{C}_{\tilde{\mathbf{X}}} \mathbf{A}^T = \mathbf{A}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix} = \mathbf{C}_{\tilde{\mathbf{Y}}\tilde{\mathbf{X}}}^T$$

- 3/ For  $n = 1$ , the density function of  $\mathbf{W}$  is just  $f_{\mathbf{X}_1}(x) = \alpha e^{-\alpha x} u(x)$ .  
 For  $n = 2$ , the density function of  $\mathbf{W}$  is just  $f_{\mathbf{X}_1+\mathbf{X}_2}(x) = f_{\mathbf{X}_1}(x) * f_{\mathbf{X}_2}(x)$ :

$$f_{\mathbf{X}_1+\mathbf{X}_2}(x) = \int_{-\infty}^{\infty} \alpha^2 e^{-\alpha y} e^{-\alpha(x-y)} u(y) u(x-y) dy = \int_0^x \alpha^2 e^{-\alpha x} dy u(x) = \alpha^2 x e^{-\alpha x} u(x).$$

Likewise for  $n = 3$ , the density function of  $\mathbf{W}$  is just  $f_{\mathbf{X}_1+\mathbf{X}_2+\mathbf{X}_3}(x) = f_{\mathbf{X}_1+\mathbf{X}_2}(x) * f_{\mathbf{X}_3}(x)$ :

$$f_{\mathbf{X}_1+\mathbf{X}_2+\mathbf{X}_3}(x) = \int_0^x \alpha^3 y e^{-\alpha y} e^{-\alpha(x-y)} dy u(x) = \frac{1}{2} \alpha^3 x^2 e^{-\alpha x} u(x).$$

For each time we convolve with another  $\alpha e^{-\alpha x} u(x)$  function, we add a factor  $\frac{\alpha}{n-1} x$  factor in going from the  $n-1$  case to the  $n$  case), so

$$f_{\mathbf{W}}(x) = \frac{\alpha^n}{(n-1)!} x^{n-1} e^{-\alpha x} u(x).$$

These are exactly the Erlang distributions.

- 4/ Let the finite value  $\mu$  denote the mean of each of the random variables, and let  $\mathbf{S}_n = \sum_{i=1}^n \mathbf{X}_i$ . Then

$$\mathcal{E}\{\mathbf{S}_n\} = \sum_{i=1}^n \mathcal{E}\{\mathbf{X}_i\} = n\mu.$$

$$\begin{aligned} \text{var}(\mathbf{S}_n) &= \mathcal{E}\{[\mathbf{S}_n - n\mu]^2\} = \mathcal{E}\{[(\sum_{i=1}^n \mathbf{X}_i) - n\mu]^2\} = \mathcal{E}\{[\sum_{i=1}^n (\mathbf{X}_i - \mu)]^2\} \\ &= \mathcal{E}\{\sum_{i=1}^n \sum_{j=1}^n (\mathbf{X}_i - \mu)(\mathbf{X}_j - \mu)\} = \sum_{i=1}^n \sum_{j=1}^n \text{cov}(\mathbf{X}_i, \mathbf{X}_j) \end{aligned}$$

Simply put,  $\text{var}(\mathbf{S}_n)$  is just the sum of all elements in the covariance matrix and since the covariance matrix is symmetric, can be the sum of the  $n$  elements on the diagonal plus double the sum of those above the diagonal.

For the first case, the diagonal elements are all  $\sigma^2$ , the  $n-1$  superdiagonal elements are all  $\rho\sigma^2$  and all other elements are zero. Thus

$$\text{var}(\mathbf{S}_n) = [n + 2(n-1)\rho]\sigma^2 = [n(1+2\rho) - 2\rho]\sigma^2.$$

For the second case, we see that again the diagonal elements are all  $\sigma^2$  and for the  $(i, j)$  element above the diagonal where  $i < j$ , the element is  $\sigma^2 \rho^{j-i}$  (which are now all nonzero but dependent only on how far off the diagonal is the element):

$$\begin{aligned} \text{var}(\mathbf{S}_n) &= n\sigma^2 + 2 \sum_{j=2}^n \sum_{i=1}^{j-1} \text{cov}(\mathbf{X}_i, \mathbf{X}_j) = n\sigma^2 + 2 \sum_{j=2}^n \sum_{l=1}^{j-1} \text{cov}(\mathbf{X}_{j-l}, \mathbf{X}_j) \\ &= \sigma^2 \{n + 2 \sum_{j=2}^n \sum_{l=1}^{j-1} \rho^l\} = \sigma^2 \{n + 2 \sum_{j=2}^n \frac{\rho - \rho^j}{1 - \rho}\} \\ &= \sigma^2 \{n + 2(n-1) \frac{\rho}{1 - \rho} - 2\rho^2 \frac{1 - \rho^{n-1}}{(1 - \rho)^2}\} \\ &= \sigma^2 \{n + 2 \frac{\rho}{1 - \rho} [(n-1) - \frac{\rho}{1 - \rho} (1 - \rho^{n-1})]\}. \end{aligned}$$

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- 5/ (a) We know from Fourier analysis that the Fourier transform of  $h(t) = e^{-\alpha t}u(t)$  is  $H(f) = [\alpha + j\omega]^{-1}$  if  $\alpha > 0$ . Then the Fourier transform of  $h(-t) = e^{\alpha t}u(-t)$  is  $H(-f) = [\alpha - j\omega]^{-1}$ . Hence the Fourier transform of  $h(t) + h(-t) = e^{-\alpha|t|}$  is

$$H(f) + H(-f) = \frac{1}{\alpha + j\omega} + \frac{1}{\alpha - j\omega} = \frac{2\alpha}{\alpha^2 + \omega^2}.$$

By Duality it follows that

$$\mathcal{F}\left\{\frac{2\alpha}{\alpha^2 + x^2}\right\} = 2\pi e^{-\alpha|\omega|},$$

and so

$$\mathcal{F}\left\{\frac{\alpha}{\pi} \frac{1}{\alpha^2 + x^2}\right\} = e^{-\alpha|\omega|}$$

The characteristic function of  $\mathbf{X}$  at  $-\omega$  is  $\mathcal{F}\{f_{\mathbf{X}}(x)\}$  so  $\Phi_{\mathbf{X}}(\omega) = e^{-\alpha|\omega|} = e^{-\alpha|\omega|}$ , giving us that  $\Phi_{\mathbf{Y}}(\omega) = e^{-\beta|\omega|}$ . Since  $\mathbf{X}$  and  $\mathbf{Y}$  are independent, the joint characteristic function is  $\Phi_{\mathbf{XY}}(\omega_1, \omega_2) = \Phi_{\mathbf{X}}(\omega_1)\Phi_{\mathbf{Y}}(\omega_2) = e^{j(\alpha|\omega_1| + \beta|\omega_2|)}$ .

- (b) The characteristic function so  $\mathbf{X} + \mathbf{Y}$  is  $\Phi_{\mathbf{XY}}(\omega, \omega) = e^{j(\alpha+\beta)|\omega|}$ , which is the characteristic function of a Cauchy random variable with parameter  $\alpha + \beta$ . Hence

$$f_{\mathbf{X}+\mathbf{Y}}(x) = \frac{\alpha + \beta}{\pi} \frac{1}{(\alpha + \beta)^2 + x^2}.$$

- (c) Applying the results of (b), we see that the density function of sum is a Cauchy random variable with parameter  $n\alpha$ , where  $\alpha$  is the parameter of the Cauchy distribution of each or the random variables  $\mathbf{X}_i$ , and then scaling this sum random variable by  $1/n$  produces another Cauchy random variable with the same distribution as each of the  $\mathbf{X}_i$ :

$$f_{\mathbf{Z}}(x) = \frac{\alpha}{\pi} \frac{1}{\alpha^2 + x^2}.$$

Clearly this is not becoming Gaussian...and is not changing at all as  $n$  increases.

- 6/ We saw stated in the notes on multiple random variables that if  $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_2, \dots, \mathbf{N}_n$  are independent identically distributed  $N(0, 1)$  random variables, then  $\mathbf{A} \triangleq \sum_{i=1}^n (\mathbf{N}_i)^2$  has a  $\chi^2$ -distribution with  $n$  degrees of freedom. Then if we have another random variable  $\mathbf{B}$  which is the sum of squares of some other  $m$  independent identically distributed  $N(0, 1)$  random variables, the sum  $\mathbf{A} + \mathbf{B}$  becomes the sum of  $n + m$  independent identically distributed  $N(0, 1)$  random variables and thus must have  $\chi^2$  distribution with  $n + m$  degrees of freedom. It follows that if  $\mathbf{A}$  and  $\mathbf{B}$  are *any* two independent  $\chi^2$  random variables with degrees of freedom  $n, m$  respectively, then  $\mathbf{A} + \mathbf{B}$  is a  $\chi^2$  random variable with  $n + m$  degrees of freedom. This then gives us that  $\mathbf{S}_n$  must be a  $\chi^2$  random variable with  $n_1 + n_2 + n_3 + \dots + n_k$  degrees of freedom.

- 7/ Let  $\mathbf{X}_i$  be a random variable describing the length of time the student would use pen  $i$ . We are given that  $\mathbf{X}_i$  has a 2-Erlang distribution with mean 9 (days). If a student buys  $N$  pens, then the time these will be used is  $\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \dots + \mathbf{X}_N$ . When all the  $\mathbf{X}_i$  are independent identically distributed 2-Erlang random variables. It follows that the exact distribution of the sum is a  $2N$ -Erlang distribution, but as  $N$  is likely to be large, this can be well approximated as having a Gaussian distribution with mean  $9N$  and variance  $N\text{var}(\mathbf{X}_i)$ . Using the mean and variance in Table 4.1, the variance of a 2-Erlang distributed random variablen with mean  $\mu$  is  $\frac{1}{2}\mu^2$ ,  $\text{var}(\mathbf{X}_i) = \sigma^2 = 40.5$ . Thus the lifetime of  $N$  pens for the student is approximately described as having a  $N(9N, 40.5N)$  distribution. Accepting this approximation as valid, the probability of  $N$  pens lasting 200 days is approximately  $Q\left(\frac{200-9N}{\sqrt{40.5N}}\right) = Q\left(\frac{(200\sqrt{2}/9)-N\sqrt{2}}{\sqrt{N}}\right)$ . For this approximation to be at least 0.95 requires  $Q\left(\frac{(200\sqrt{2}/9)-N\sqrt{2}}{\sqrt{N}}\right) \geq 0.95$ , which requires

$$\begin{aligned} \frac{(200\sqrt{2}/9)-N\sqrt{2}}{\sqrt{N}} \leq -1.645 &\implies \sqrt{2}(\sqrt{N})^2 - 1.645\sqrt{N} - \frac{200\sqrt{2}}{9} \geq 0 \\ \implies \sqrt{N} &\geq \frac{1.645 + \sqrt{(1.645)^2 + 1600/9}}{2\sqrt{2}} \simeq 5.331. \end{aligned}$$

Thus  $N \geq 28.424$ , or more plainly, the student needs to buy at least 29 pens. The approximate result does depend on the exact distribution of the lifetime of one pen, but only in the relationship between mean and variance. it is not to hard to solve it again for a  $\chi^2$ -distribution for the lifetime of a pen with the same approach:

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7/ (continued)

When all the  $\mathbf{X}_i$  are independent identically distributed Rayleigh random variables, it is again hard to find the exact distribution of the sum, but as  $N$  is likely to be large, this can be well approximated as having a Gaussian distribution with mean  $9N$  and variance  $N\text{var}(\mathbf{X}_i)$ . Using the mean and variance in Table 4.1, the variance of an Rayleigh distribution with mean  $\mu$  is  $(\frac{4}{\pi}-1)\mu^2$ ,  $\text{var}(\mathbf{X}_i) = \sigma^2 = (\frac{4}{\pi}-1) \times 81 \simeq 22.1324$ . Thus the lifetime of  $N$  pens for the student is approximately described as having a  $N(9N, 22.1324N)$  distribution. Accepting this approximation as valid, the probability of  $N$  pens lasting 200 days is approximately  $Q(\frac{200-9N}{\sqrt{81(\frac{4}{\pi}-1)N}}) = Q(\frac{(200/9)-N}{\sqrt{(\frac{4}{\pi}-1)N}})$ . for this approximation to be at least 0.95 requires  $Q(\frac{(200/9)-N}{\sqrt{(\frac{4}{\pi}-1)N}}) \geq 0.95$ , which requires

$$\begin{aligned} \frac{(200/9)-N}{\sqrt{(\frac{4}{\pi}-1)N}} \leq -1.645 &\implies (\sqrt{N})^2 - 1.645\sqrt{(\frac{4}{\pi}-1)}\sqrt{N} - \frac{200}{9} \geq 0 \\ \implies \sqrt{N} &\geq \frac{0.44948 + \sqrt{(0.44948)^2 + 800/9}}{2} \simeq 4.944. \end{aligned}$$

Thus  $N \geq 24.4445$ , or more plainly, the student needs to buy at least 25 pens.

8/ If  $\mathbf{X} \sim U(-\frac{1}{2}, \frac{1}{2})$ ,  $\mathcal{E}\{\mathbf{X}\} = 0$  and  $\text{var}(\mathbf{X}) = \frac{1}{12}$ . Hence the error in adding 128 numbers together has zero mean and variance  $128/12 = 32/3$  (assuming the errors in each number are uncorrelated with respect to each other). Hence appealing to the CLT, the total error is well-approximated as having a distribution  $N(0, 32/3)$  and so

$$\mathcal{P}(|\text{error}| > 4.5) \simeq 2Q(4.5/\sqrt{32/3}) \simeq 2Q(1.37784) \simeq 2(0.784Q(1.38) + 0.216Q(1.37)) \simeq 0.1682.$$

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