Université d'Ottawa
Faculté de génie

École de science informatique
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University of Ottawa
Faculty of Engineering

School of Electrical Engineering
and Computer Science

ELG 3126 **RANDOM SIGNALS AND SYSTEMS** **Winter 2023**

ASSIGNMENT 12

(not to be handed in)

1. A coin of unknown character is tossed independently 800 times, and it is found that 413 times the result was "Heads". Find the 98% confidence interval for the estimate of the probability of a "Heads" using the fraction of times a "Heads" occurred as the estimate.
2. From the text: 8.18(a)(b)(d). [Do not derive the results in Prob. 8.10, just use them.]
3. From the text: 8.40, but use probability 90% in (a) and 99.9% in (b).
4. From the text: 8.43, but just find 95% confidence intervals.
5. In a certain radar situation, the receiver produces an output which is described by a random variable $\mathbf{R}$. When there is no target, the distribution that $\mathbf{R}$ has is a zero mean Gaussian random variable with variance 1. When a target is present, the $\mathbf{R}$ has the distribution

$$f_{\mathbf{R}}(r) = \tfrac{1}{16}|r|e^{-r^2/16}.$$

The radar receiver must use $\mathbf{R}$ to decide between

$$\begin{aligned} H_0 &: \text{ no target is present;} \\ H_1 &: \text{ a target is present.} \end{aligned}$$

The operation of the radar receiver is such that the receiver will decide that a target is present when $|\mathbf{R}| > 2$, and is otherwise will decide the target is absent.

(a) What is the probability of a false alarm (the probability a target is said to be present when there is none)? What is the level of significance of the test the radar receiver uses?

(b) What is the probability of a missed detection (the probability that the radar says there is no target when one actually is present)?

6. From the text: 8.70(a)
7. From the text: 8.94
8. Five thumbtacks of a certain type were "tossed" 200 times and the number of thumbtacks landing pointing up was counted each time. The results were:

   on 5 tries, no tack pointed up;
   on 25 tries, 1 tack pointed up;
   on 41 tries, 2 tacks pointed up;
   on 68 tries, 3 tacks pointed up;
   on 44 tries, 4 tacks pointed up;
   on 17 tries, 5 tacks pointed up.

(a) Find an estimate (with a 95% confidence interval) for the probability p that a thumbtack falls point up.
Hint: There are 1000 total throws of a tack involved here. We may assume they are tossed so that each is an independent Bernoulli trial, just as in the example of counting errors in a binary digital communications system described by a binary symmetric channel model.

8. [*continued*]

   (b) Use the χ^2-goodness of fit test to test the hypothesis that the number of tacks that fall pointing up has a binomial distribution for $n = 5$. To estimate the value of p, use the result from (a). Use a 5% level of significance.

9. Suppose we have 20 random samples of a random quantity and find that this sample has a sample mean of 10.21, and a sample variance of 1.95.

   (a) If it is known that variance of the population is 2 and the samples are all independent and follow a Gaussian distribution, what is the 90% confidence interval the mean of the quantity in question? What is the 95% confidence interval?

   (b) If it is known that the samples are all independent and follow a Gaussian distribution but the variance is unknown, what is the 90% confidence interval the mean of the quantity in question? What is the 95% confidence interval?

   (c) If it is known that the samples are all independent and follow a Gaussian distribution but the variance is unknown, what is the 90% confidence interval the variance of the quantity in question? What is the 95% confidence interval?

10. You have been sent an EXCEL file that gives the results of the coin tossing done in the lab for each of the 18 initial groups in this years's ELG3126 class.

   (a) Compute the sample mean and sample variance of the numbers of heads your group observed for the number of heads on a coin toss. Repeat this for each set of data. Do NOT use the AVERAGE and STDEV (or STDEVA, STDEVP, STDEVPA, STDEV.P or STDEV.S) functions to begin with (you may use the SUM function to add a list of numbers), but then repeat the calculations using them and determine whether they function as you would expect them to. (A sample standard deviation is just the square root of the sample variance we discussed in class.) Some of these may apply to EXCEL in MS Office 2011 and later and not to MS Office 2008 and earlier

   (b) Explain the differences between STDEV, STDEVA, STDEVP, STDEV.P and STDEV.S. Consult the description of these in the help files on your system.

   (c) If we assume that the sample mean in (a) is well-described as being a Gaussian random variable with mean p, what is the 92% confidence interval for p for the last year group with your group number or the number closest to yours if there is no exact fit. Repeat this calculation if we assume that the results from all the groups listed in the table for all years represents repeated tosses of the same coin and so use the combined list of coin toss results as the basis for estimating p.

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ASSIGNMENT 12

SOLUTIONS

1/ From the discussion in class, we know that the $(1-\alpha)\%$ confidence interval for the probability of a "Heads", p, is reasonably well estimated as $[\hat{p}-\lambda\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}, \hat{p}-\lambda\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}]$, where $\hat{p}$ is the fraction of the N tosses that turned up "Heads", and λ is that value where $Q(\lambda) = \frac{1}{2}(\alpha)$ (assuming Np is moderately large). In this problem $\alpha = 0.02$, $\hat{p} = 0.51625$, $N = 800$ and $\lambda = 2.32634$ (we can get this value approximately from the Q-function table, or more precisely as given here from the last line in the Student-t distribution table as the Student-t distribution in the limit of the degrees of freedom becoming arbitrarily large; since our confidence interval here is an estimate, there is no real use for extreme precision). This evaluates to $\underline{\underline{[0.475, 0.557].}}$

2/ (8.18(a)(b)(d))
For a random variable $\mathbf{X} \sim U(0,\theta)$,

$$f_{\mathbf{X}}(x) = \begin{cases} \frac{1}{\theta}, & \text{if } 0 < x < \theta; \\ 0, & \text{otherwise;} \end{cases} \qquad F_{\mathbf{X}}(x) = \begin{cases} 1, & \text{if } x \geq \theta; \\ \frac{x}{\theta}, & \text{if } 0 < x < \theta; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Using the above and the results of Question 8.10, we have that

$$f_{\hat{\boldsymbol{\theta}}}(x) = \begin{cases} \frac{nx^{n-1}}{\theta^n}, & \text{if } 0 < x < \theta; \\ 0, & \text{otherwise.} \end{cases}$$

(b) The mean value of the estimator random variable $\hat{\boldsymbol{\theta}}$ is

$$\mathcal{E}\{\hat{\boldsymbol{\theta}}\} = \int_{-\infty}^{\infty} x f_{\hat{\boldsymbol{\theta}}}(x)\,dx = \int_0^{\theta} x f_{\hat{\boldsymbol{\theta}}}(x)\,dx = \frac{n}{\theta^n}\int_0^{\theta} x^n\,dx = \frac{n}{\theta^n}\frac{\theta^{n+1}}{n+1} = \frac{n}{n+1}\theta.$$

As this is not θ, we know that this estimator is *biased*.

Remark: The estimator here is the value for θ that make the observed value most likely, and thus is term the maximum likelihood estimator of θ. Choosing the value of a parameter that make what is observed the most likely to occur is a commonly used approach to deciding on an estimator, and the result is known as the *maximum likelihood estimator*. As we see here, such an estimator is not necessarily unbiased. In the problem of estimating a population mean from independent Gaussian samples, the sample mean is the maximum likelihood estimator of the mean.

(d) Clearly to make the estimator unbiased, me should simply scale $\hat{\boldsymbol{\theta}}$ by $c = (n+1)/n$.

3/ (8.40)-modified

In this problem, the estimator being used is presumably the sample mean. Since the variance of each sample random variable $\mathbf{X}_i$ is 1, and these sample random variable are independent and thus uncorrelated, the estimator random variable is a Gaussian random variable with mean μ and variance $1/n$. Thus the confidence interval is $[\mu - \lambda/\sqrt{n}, \mu + \lambda/\sqrt{n}]$, where λ is determined by the requirement that $Q(\lambda) = \frac{1}{2}\alpha$ where $(1-\alpha)$ is the desired confidence level as a probability. The width of the confidence interval is $2\lambda/\sqrt{n}$.

(a) For a 90% confidence, from the last line of the Student t distribution table distributed in class (the 0.05 column), $\lambda \simeq 1.64485$. Thus the confidence interval width for the various stated n is:

for $n = 4$, the width is approximately $2 \times 1.64485/2 = \underline{\underline{1.64485}}$;

for $n = 16$, the width is approximately $2 \times 1.64485/4 = \underline{\underline{0.82243}}$;

for $n = 100$, the width is approximately $2 \times 1.64485/10 = \underline{\underline{0.32897}}$.

(b) For a 99.9% confidence, from the Q-function table, $\lambda \simeq 3.29$. The more exact answer can be found using Excel or MATLAB, $\lambda \simeq 3.29053$ Thus the confidence interval width for the various stated n is:

for $n = 4$, the width is approximately $2 \times 3.29053/2 = \underline{\underline{3.29053}}$;

for $n = 16$, the width is approximately $2 \times 3.29053/4 = \underline{\underline{1.64526}}$;

for $n = 100$, the width is approximately $2 \times 3.29053/10 = \underline{\underline{0.65811}}$.

4/ (8.43)

(a) This is a straightforward application of the special case of estimating a mean of a Gaussian population with unknown mean and variance: the confidence interval is $[\mu_s - \lambda\sqrt{\frac{s^2}{n}}, \mu_s + \lambda\sqrt{\frac{s^2}{n}}]$, where λ is the value where the cdf of the Student t distribution with $n-1$ degrees of freedom has value $1 - \frac{1}{2}(1-\alpha)$ for confidence α. From the Student t tables for 9 degrees of freedom:

for 90%, $\lambda = 1.83311 \implies \underline{\underline{[54.51, 60.09]}}$

for 95%, $\lambda = 2.26215 \implies \underline{\underline{[53.85, 60.75]}}$

for 99%, $\lambda = 3.24984 \implies \underline{\underline{[52.35, 62.25]}}$

(b) This only changes the number of degrees of freedom from 9 to 19. From the Student t tables for 19 degrees of freedom:

for 90%, $\lambda = 1.72913 \implies \underline{\underline{[54.67, 59.93]}}$

for 95%, $\lambda = 2.09302 \implies \underline{\underline{[54.11, 60.49]}}$

for 99%, $\lambda = 2.86094 \implies \underline{\underline{[52.94, 61.66]}}$

(c) Now we base our confidence interval on the basis that $s^2(n-1)/\sigma^2$ has a χ^2 distribution with $(n-1)$ degrees of freedom (per the discussion in class): the confidence interval is $[s^2(n-1)/\lambda_2, s^2(n-1)/\lambda_1]$, where λ_1 and λ_2 come from the χ^2 tables for $n-1$ degrees of freedom.

For $n = 10$ (a):

for 90%, $\lambda_1 = 3.32512,\ \lambda_2 = 16.91896 \implies \underline{\underline{[12.34, 62.79]}}$

for 95%, $\lambda_1 = 2.70039,\ \lambda_2 = 19.02278 \implies \underline{\underline{[10.98, 77.32]}}$

for 99%, $\lambda_1 = 1.73491,\ \lambda_2 = 23.58927 \implies \underline{\underline{[8.85, 120.35]}}$

For $n = 20$ (b):

for 90%, $\lambda_1 = 10.1170\ \lambda_2 = 30.1435 \implies \underline{\underline{[6.93, 20.64]}}$

for 95%, $\lambda_1 = 8.9065,\ \lambda_2 = 32.8523 \implies \underline{\underline{[6.36, 23.44]}}$

for 99%, $\lambda_1 = 6.8440,\ \lambda_2 = 38.5823 \implies \underline{\underline{[5.41, 30.51]}}$

5/ (a) The probability of false alarm is $\mathcal{P}(|\mathbf{R}| > 2 \mid H_0)$ which is

$$P_F = 2\mathcal{P}(\mathbf{R} > 2 \mid H_0) = 2Q(2) \simeq 0.04550 \qquad \text{[since under } H_0,\ \mathbf{R} \sim N(0,1)\text{]}.$$

This is the probability of a Type I error and so is the level of significance of the test (about 4.55%).
(b) The probability of a missed detection is

$$P_M = \mathcal{P}(\mathbf{R} < 2 \mid H_1) = \int_{-2}^{2} \tfrac{1}{16}|r|e^{-r^2/16}\,dr = \tfrac{1}{8}\int_0^2 re^{-r^2/16}\,dr = -\int_0^2 \frac{d}{dr}e^{-r^2/16}\,dr = 1 - e^{-1/4} \simeq \underline{\underline{0.22120}}.$$

6/ (8.70)(a)
We are interested in this problem in determining whether the hypothesis that the mean battery life in a computer is 4 hours (H_0) or some lower value (H_1) based on a sample mean of 100 samples, μ_s, and using the sample mean as the deciding factor. The test to adopt in this case is a one-sided case where if $\mu_s < \lambda$, we accept the hypothesis H_0, while if $\mu_s \geq \lambda$, we accept the hypothesis H_1. If we assume that the time a battery charge lasts is influenced by many factors, then the random variable describing this charge duration for a computer should be well-described as having a Gaussian distribution with some mean μ and variance σ^2. Assuming this is the case, then we know that the statistic $\mathbf{t} = (\boldsymbol{\mu}_s - \mu)/\sqrt{\boldsymbol{\sigma}_s^2/n}$ has a Student-t distribution with $(n-1)$ degrees of freedom. Thus we can use the t distribution for $(n-1)$ degrees of freedom to find the value of z so that $\mathcal{P}(\mathbf{t} > z) = \alpha$. For 99 degrees of freedom and $\alpha = 0.01$, $z = 2.6264$, and for $\alpha = 0.05$, $z = 1.9842$. Thus if H_0 was valid, we have

$$\mathcal{P}(\boldsymbol{\mu} > 4 - 2.6264 * \tfrac{1}{2}/10) = \mathcal{P}(\boldsymbol{\mu} > 3.87) = 0.01.$$

Hence we have our test for $\alpha = 0.01$: if the sample mean is greater than 3.87 hours, we accept that the evidence supporst the hypothesis that the average charge life is 4 hours (and therwise we reject the hypothesis). As 3.0 is below the 3.87 threshold, we reject the hypothesis that the battery life is 4 hours at the 1% level of significance.

If we repeat the above for a 5% level of significance, the threshold becomes $4 - 1.9842 * \frac{1}{2}/10 = 3.90$, and again the hypothesis is rejected (at the 5% level of significance).

7/ (8.94)
The table for the calculation of the χ^2-goodness of fit test for the $\{0, 1, 2, \ldots, 9\}$ is

Digit	Observed No. of Samples, f_i	Expected No. of Samples, $n\tilde{p}_i$	$(f_i - n\tilde{p}_i)^2$	$\frac{(f_i - n\tilde{p}_i)^2}{n\tilde{p}_i}$
0	0	10.5	110.25	10.5
1	0	10.5	110.25	10.5
2	24	10.5	182.25	17.36
3	2	10.5	72.25	6.88
4	25	10.5	210.25	20.02
5	3	10.5	56.25	5.36
6	32	10.5	462.25	44.02
7	15	10.5	20.25	20.25
8	2	10.5	72.25	72.25
9	2	10.5	72.25	6.88
	105	105		130.33

This statistic should be described (approximately) as having a χ^2 distribution with 9 degrees of freedom, and so for a 1% level of significance, the threshold is given by 21.660. Since 130.33 > 21.660, we find that this data does not support the hypothesis that digits 0 thought 9 occur with uniform probability at the 1% level of significance. *The hypothesis is rejected at the 1% level of significance.*

7/ (continued)
For the set of digits 2 through 9, the new calculation of the χ^2 statistic is provided in the table below:

Digit	Observed No. of Samples, f_i	Expected No. of Samples, $n\tilde{p}_i$	$(f_i - n\tilde{p}_i)^2$	$\frac{(f_i - n\tilde{p}_i)^2}{n\tilde{p}_i}$
2	24	13.125	118.27	9.01
3	2	13.125	123.77	9.43
4	25	13.125	141.02	10.74
5	3	13.125	102.52	7.81
6	32	13.125	356.27	27.14
7	15	13.125	3.52	0.27
8	2	13.125	123.77	9.43
9	2	13.125	123.77	9.43
	105	105		83.27

This statistic should be described (approximately) as having a χ^2 distribution with 7 degrees of freedom, and so for a 1% level of significance, the threshold is given by 18.4753. Since $83.27 > 18.4753$, we find that this data does not support the hypothesis that digits 2 thought 9 occur with uniform probability at the 1% level of significance. *The hypothesis is rejected at the 1% level of significance.*

8/ (a) For an estimate of p we use the fraction of the 1000 thumb tack tosses that lands pointing up:

$$\hat{p} = (0 \times 5 + 1 \times 25 + 2 \times 41 + 3 \times 68 + 4 \times 44 + 5 \times 17)/1000 = 0.572.$$

As discussed in class, the 95% confidence interval would then be approximately

$$[\hat{p} - 1.96\sqrt{\hat{p}(1-\hat{p})/1000}, \hat{p} - 1.96\sqrt{\hat{p}(1-\hat{p})/1000}] \simeq \underline{\underline{[0.541, 0.603]}}$$

(b) The distribution to compare with is the binomial distribution $B(5, 0.572)$. The probability then of getting k tacks pointing up (for $k \in \{0, 1, 2, 3, 4, 5, 6\}$) is

$$p_k = \binom{5}{k}\hat{p}^k(1-\hat{p})^{5-k} = \binom{5}{k}(0.572)^k(0.428)^{5-k}.$$

These values are $p_0 = 0.014362$, $p_1 = 0.0959712$, $p_2 = 0.256521$, $p_3 = 0.3428275$, $p_4 = 0.2290857$, $p_5 = 0.061232$. The expected number of observations of k tacks pointing up in 200 tosses of the five tacks is $200p_k$:

$$200p_0 = 2.872,\ 200p_1 = 19.194,\ 200p_2 = 51.304,\ 200p_3 = 68.565,\ 200p_4 = 45.817,\ 200p_5 = 12.246.$$

To apply the χ^2 Goodness of Fit test will require we group the $k = 0$ bin with another so that each bin has at east 5 expected elements. If we group $k = 0$ and $k = 1$ bins together, the computations for the χ^2 Goodness of Fit test is as follows ($\tilde{p}_i$ is the probability of being in the ith bin):

Bin i	f_i	$\tilde{p}_i$	$n\tilde{p}_i$	$(f_i - n\tilde{p}_i)^2$	$\frac{(f_i - n\tilde{p}_i)^2}{n\tilde{p}_i}$
$k = 0$ or 1	30	0.11033	22.0667	62.9377	2.852
$k = 2$	41	0.25652	51.3043	106.1776	2.070
$k = 3$	68	0.34282	68.5655	0.3198	0.005
$k = 4$	44	0.22909	45.8171	3.3020	0.072
$k = 5$	17	0.06123	12.2464	22.5963	1.845
					Total: 6.844

8/ (continued)

Since 5 bins are used and we estimated one parameter of the hypothetical distribution, the test statistic is well described by a χ^2 distribution with three degrees of freedom. For a 5% level of significance, the threshold is 9.348. Since our value of the statistic of 6.843 is less than the threshold, we accept the hypothesis that the count of thumb tacks pointing up is governed by a binomial distribution—at the 5% level of significance. Put another way, we can say *the observations are consistent with a binomial distribution model at the 5% level of significance.*

9/ (a) The 90% confidence interval is $[\mu_s - \frac{1.845\sigma}{\sqrt{n}}, \mu_s + \frac{1.845\sigma}{\sqrt{n}}$, where here $\mu_s = 10.21$, $\sigma^2 = 2$, and $n = 20$. This evaluates to $[9.69, 10.73]$. The 95% confidence interval is $[\mu_s - \frac{1.96\sigma}{\sqrt{n}}, \mu_s + \frac{1.96\sigma}{\sqrt{n}}$. This evaluates to $\underline{\underline{[9.59, 10.83]}}$.

(b) As per the analysis in class, we use the Student t-distribution with $n - 1 = 19$ degrees of freedom to establish a critical value and use s^2 in place of σ^2. For a 90% confidence interval, the critical value for 19 degrees of freedom is 1.729, so the 90% confidence intervals is $[\mu_s - 1.729\sqrt{\frac{s^2}{20}}, \mu_s + 1.729\sqrt{\frac{s^2}{20}}$ which is $\underline{\underline{[9.67, 10.75]}}$ For a 95% confidence interval, simply change 1.729 to 2.093, giving the interval $\underline{\underline{[9.56, 10.86]}}$.

(c) Following the analysis in class, for a 90% confidence interval, we need the values λ_1 and λ_2 at which the χ^2 distribution with 19 degrees of freedom have cumulative probabilities 0.05 and 0.95, which are 10.117 and 30.149 giving a confidence interval for σ^2 to be $[\frac{(n-1)s^2}{\lambda_2}, \frac{(n-1)s^2}{\lambda_1}$ which is $\underline{\underline{[1.23, 3.66]}}$. For a 95% confidence interval: $\lambda_1 = 8.907$ and $\lambda_2 = 32852$, giving the interval $\underline{\underline{[1.13, 4.16]}}$.

10/ (a) The sample mean and sample variance for each group's measurements and for all the measurements combined are given below for the data from Spring 2012. You will report your results for the Winter 2023 groups in the lab report. "Group −1" and "Group 0" are not real groups so we give the results for all the groups including and excluding those results. These results are generated using the AVERAGE and STDEV functions in Excel.

	Group −1	Group 0	Group 1	Group 2	Group 3	Group 4	Group 5	All Groups	All Real Groups
Sample Mean	0.5000	0.5000	0.5000	0.5400	0.4700	0.4700	0.5800	0.5025	0.5030
Sample Variance	0.2525	0.2525	0.2525	0.2509	0.2516	0.2516	0.2461	0.2502	0.2502

	Group 6	Group 7	Group 8	Group 9	Group 10
Sample Mean	0.4700	0.4700	0.4600	0.5200	0.5500
Sample Variance	0.2516	0.2516	0.2509	0.2521	0.2500

Fig. 10.1. Sample Mean and Variance for 100 Coin Tosses, Spring 2012 class.

(b) The differences between STDEV and STDEVP are in the division of $\sum(x_i - \mu_s)^2$ by $(n-1)$ or n to provide the estimate of the population variance. As we showed in class, for a set of n randomly selected samples, $(n-1)$ is appropriate and produces an unbiased estimate of the population variance σ^2. When the samples are the entire population, selected so that each element is chosen once, then division by n is appropriate and this gives the actual population variance, not an estimate. The Excel function STDEVA and STDEVP are functions that allow other sorts of arguments besides numerical sample values. STDEV.S and STDEV.P are new versions of STDEV and STDEVP that may be more numerically accurate in some cases.

(c) In class we considered the problem of calculating the confidence interval for the probability of "success" p estimated by $\hat{p} = X/n$, where X is the number of observed "successes" after n Bernoulli trials. We observed that the random variable describing the number of "successes", $\mathbf{X}$, has a binomial distribution $B(n, p)$ with a mean np and variance $np(1-p)$, so that the random variable $\hat{\mathbf{p}} = \mathbf{X}/n$ has mean p and variance $p(1-p)/n$. When n is large, we argued that the distribution of the error could be well-approximated, then, as a zero-mean Gaussian random variable with variance $p(1-p)/n$, which provided that np is large (greater than 10), would be well approximated as a variance $\hat{p}(1-\hat{p})/n$. This gave as that the confidence interval can (when a large number of "successes" has been found) be *approximated* as

$$[\ \hat{p} - \lambda\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + \lambda\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\ ],$$

10/(c) (continued)

where λ is set according to the desired level of confidence $1-\alpha$ in that $Q(\lambda) = \frac{1}{2}\alpha$ ($\lambda = 1.960$ for a 95% level of confidence, $\lambda = 1.645$ for a 90% confidence and $\lambda = 1.150$ for a 75% confidence).

All of this can be applied directly to the probability of a "Heads" on a coin toss as p, the probability of a "Heads". Then since the value for np here is about 50, the 75% confidence interval is *approximated* to be

$$[\ \hat{p} - 1.15\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 1.15\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\ ],$$

This produces the following (estimated) confidence intervals for the data for each group in the Spring 2012 class:

	Group −1	Group 0	Group 1	Group 2	Group 3	Group 4	Group 5
lower limit	0.4425	0.4425	0.4425	0.4827	0.4126	0.4126	0.5232
upper limit	0.5575	0.5575	0.5575	0.5973	0.5274	0.5274	0.6368
			Group 6	Group 7	0.3015	Group 9	Group 10
			0.4126	0.4126	0.4027	0.4625	0.4928
			0.5274	0.5274	0.5173	0.5775	0.6072
CONFIDENCE LE\	0.75	1.1503	(THRESHOLD LAMBDA)				

Fig. 10.2. Upper and lower limits of the estimated 75% confidence interval for $\mathcal{P}$(Heads) for each Spring 2012 group.

The above approach to estimating a confidence interval was based on the knowledge that the coin tosses were each independent experiments. If this were not the case, or we were not that certain the Gaussian approximations we made were valid, we might resort to using the method of batched means described in the text. For this, we might divide the 100 coin tosses into 10 groups of 10 tosses, and produce an estimate for p form each of the 10 sets of 10 tosses, which would be a set of 10 values described by 10 Gaussian random variables with mean p and some unknown variance. The 10 numbers would more likely be uncorrelated one with respect to another. To produce an estimate of the confidence interval we could then simply apply the results on estimating the mean of a Gaussian population with unknown variance (thus using the Student-t distribution). Of course the average of the 10 averages is just the average of all 100 numbers, so the basic estimate does not change, just the estimate of the confidence interval.

Method of batched means												
est 1	1	0.5	0.6	0.8	0.3	0.5	0.5	0.5	0.5	0.4	0.5	0.7
	1	0.4	0.4	0.6	0.1	0.7	0.5	0.5	0.3	0.3	0.5	0.5
	1	0.6	0.5	0.4	0.6	0.4	0.9	0.2	0.4	0.4	0.5	0.6
	1	0.4	0.7	0.4	0.6	0.5	0.4	0.5	0.6	0.5	0.3	0.5
	1	0.6	0.4	0.5	0.5	0.3	0.6	0.6	0.8	0.4	0.6	0.5
	0	0.4	0.5	0.5	0.5	0.5	0.5	0.4	0.3	0.4	0.7	0.7
	0	0.6	0.4	0.4	0.2	0.3	0.5	0.5	0.3	0.6	0.3	0.5
	0	0.4	0.5	0.7	0.6	0.4	0.7	0.3	0.7	0.6	0.8	0.7
	0	0.6	0.6	0.4	0.7	0.7	0.7	0.6	0.4	0.4	0.6	0.3
est 10	0	0.5	0.4	0.7	0.6	0.4	0.5	0.6	0.4	0.6	0.4	0.5
average	0.5	0.5	0.5	0.54	0.47	0.47	0.58	0.47	0.47	0.46	0.52	0.55
sample stndev	0.527	0.0943	0.1054	0.1506	0.2003	0.1418	0.1476	0.1337	0.1767	0.1075	0.1619	0.1269

	Group −1	Group 0	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	Group 7	Group 8	Group 9	Group 10
lower limit	0.2181	0.4496	0.4436	0.4595	0.3629	0.3941	0.5011	0.3985	0.3755	0.4025	0.4334	0.4821
upper limit	0.7819	0.5504	0.5564	0.6205	0.5771	0.5459	0.6589	0.5415	0.5645	0.5175	0.6066	0.6179

Fig. 10.3. Upper and lower limits of the estimated 75% confidence interval for $\mathcal{P}$(Heads) using the batched means method (10 groups of 10)