Université d'Ottawa
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et de génie électrique

University of Ottawa
Faculty of Engineering

School of Electrical Engineering
and Computer Science

uOttawa
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Canada's university

ELG 3125
Signal and System Analysis

Final Exam

Time allowed: 3 hours Sunday, Dec. 17, 2021, 2:00-5:00 pm

Instructor: Jianping Yao

Last name: ______________________________

First name: ______________________________

Student number: __________________________

Signature: _______________________________

Q1	/25
Q2	/15
Q3	/20
Q4	/20
Q5	/20
Total	/100

- Closed-book exam;
- No programmable calculators are permitted;
- Formula sheets are provided in the last pages of this exam.

Question 1 Continuous-time Fourier transform, continuous time LTI systems /25

Q1.1 A causal and stable LTI system has the following frequency response,

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3}.$$

a) Find the impulse response $h(t)$ of that system. (/5)

b) If an input $x(t) = te^{-2t}u(t)$ is applied to the system, find the output signal $y(t)$. (/5)

c) Find a differential equation describing the relation between the input $x(t)$ and the output $y(t)$ of that LTI system. (/5)

Q1.2 Find the total energy (i.e., $\int_{t=-\infty}^{\infty} |x(t)|^2 \, dt$) of the following signal: $x(t) = \dfrac{\sin(20t)}{\pi t}$. The use of property tables is recommended. (/5)

Q1.3 For a given stable LTI system, if the input is $x(t) = e^{-2t}u(t)$, the output is $y(t) = e^{-3t}u(t) - e^{4t}u(-t)$. Find the frequency response $H(j\omega)$ and the impulse response $h(t)$ of that system. (/2+3)

Question 2 Discrete-time Fourier transform **/15**

Consider a discrete time LTI system with the following impulse response: $h[n]=\left(\frac{1}{2}\right)^n u[n]+\frac{1}{2}\left(\frac{1}{4}\right)^n u[n]$.

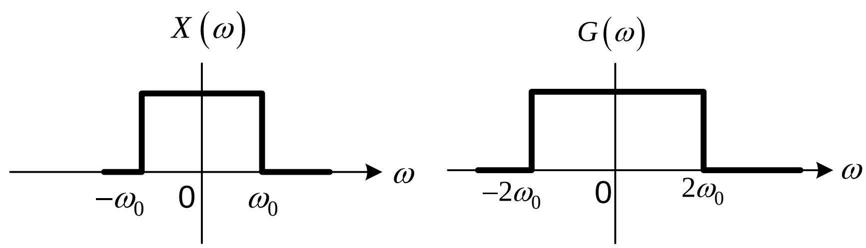
a) Determine the difference equation relating the input $x[n]$ and the output $y[n]$ of that system. (/10)

b) Determine the output $y[n]$ of the system if the input is $x[n]=\delta[n]+\delta[n-5]$. (/5)

Question 3 Sampling **/20**

Q3.1 Let $x(t)$ and $g(t)$ be two real-valued baseband signals with their maximum frequencies being ω_0 and $2\omega_0$, respectively. Determine the minimum sampling rate required to sample the following signals without loss of information, i.e., the Nyquist rate:

a) $x(t) * g(t) + 3x(t)$ (/5)
b) $x(t) \times g(t) + 3x(t)$ (/5)



Q3.2 A continuous-time signal $x_c(t) = \left(\frac{\sin 100\pi t}{\pi t}\right)^2$ is uniformly sampled with a sampling period $T = \frac{1}{f_s} = \frac{2\pi}{\omega_s}$, to produce a discrete-time signal $x_p(t)$, with its Fourier transform being denoted by $X_p(j\omega)$.

a) For a sampling frequency $\omega_s = 100\pi$, plot $X_p(j\omega)$. (/5)

b) Is it possible to reconstruct $x_c(t)$ from the sampled signal $x_p(j\omega)$, using a lowpass filter? Explain. (/5)

Question 4 Bode plot and filtering **/20**

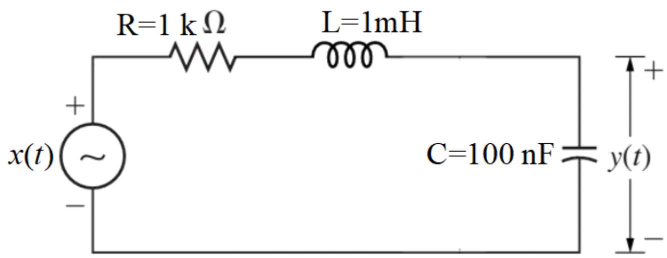
Q4.1 The following system can be modeled as a second-order LTI system with the standard form:

a) Write the differential equation that characterizes the system (using *R*, *L*, *C*, with no numerical values in the expression) (**/3**)

b) Write the frequency response of the system (using *R*, *L*, *C*, with no numerical values) (**/3**)

c) Determine if the system is under, critically or over damped. Determine if the impulse response $h(t)$ of the system contains oscillations or not. (**/2**)

d) If the value of R can be tuned, determine the value of R for the system that is critically damped. (**/2**)



Note: $x(t)$ is the input voltage (in Volts), and $y(t)$ is the output voltage over the capacitor (in Volts).

Q4.2 An LTI system has the following transfer function:

$$H(j\omega) = \frac{10^{12}(j\omega + 100)}{(j\omega + 1000)\,((j\omega)^2 + 9000(j\omega) + 10^8)}.$$

a) Use straight line approximation, draw the Bode plot of the **magnitude** response. (**/4**)
b) What type of filter is it? (**/1**)

Question 5 Laplace Transform and LTI System Analysis **/20**

Q5.1 Consider a continuous-time LTI system for which the input $x(t)$ and $y(t)$ are related by the following differential equation:

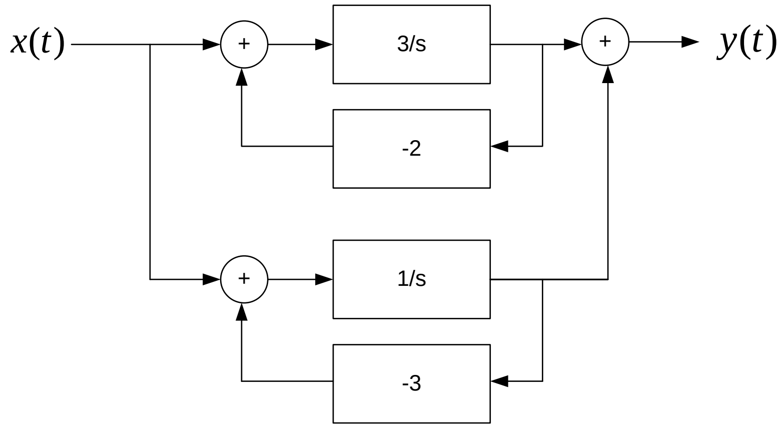
$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 6y(t) = x(t).$$

a) Determine the system's transfer function $H(s)$. (/5)

b) Sketch the pole-zero pattern of $H(s)$ in the *s*-plane. (/5)

c) Determine $h(t)$ for each of the following cases: (i) The system is stable. (ii) The system is causal. (iii) The system is neither stable nor causal. (/6)

Q5.2 A causal LTI system S has the block diagram representation shown in the figure below. Determine the differential equation relating the input $x(t)$ and output $y(t)$ of the system. (/4)



A few formulas

Euler

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Summations, geometric series

$$\sum_{k=n_1}^{\infty} a^k = \frac{a^{n_1}}{1-a} \quad |a| < 1 \qquad \sum_{k=n_1}^{n_2} a^k = \frac{a^{n_1} - a^{n_2+1}}{1-a} \quad a \neq 1,\ n_2 \geq n_1$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \quad |a| < 1 \qquad \sum_{k=0}^{n_1} a^k = \frac{1-a^{n_1+1}}{1-a} \quad a \neq 1,\ n_1 \geq 0$$

Even and odd parts

$$x_e(t) = \frac{1}{2}x(t) + \frac{1}{2}x(-t) \qquad x_o(t) = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$$

$$x_e[n] = \frac{1}{2}x[n] + \frac{1}{2}x[-n] \qquad x_o[n] = \frac{1}{2}x[n] - \frac{1}{2}x[-n]$$

Convolutions

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$h(t)$ response for differential equations describing LTI systems (single order roots)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$h(t) = \sum_{k=0}^{N-1} A_k e^{s_k t} u(t) + \sum_{k=0}^{M-N} B_k \frac{d^k \delta(t)}{dt^k}$$

$h[n]$ response for difference equations describing LTI systems (single order roots)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$h[n] = \sum_{k=0}^{N-1} A_k \alpha_k^n u[n] + \sum_{k=0}^{M-N} B_k \delta[n-k]$$

LTI systems and eigenfunctions

$$e^{st} \xrightarrow{LTI(cont.)} H(s)e^{st}$$

$$z^n \xrightarrow{LTI(discr.)} H(z)z^n$$

$$e^{j\omega t} \xrightarrow{LTI(cont.)} H(j\omega)e^{j\omega t}$$

$$e^{j\omega n} \xrightarrow{LTI(discr.)} H(e^{j\omega})e^{j\omega n}$$

$$\cos(\omega t) \xrightarrow{LTI(cont.)} |H(j\omega)|\cos(\omega t + \measuredangle H(j\omega))$$

$$\cos(\omega n) \xrightarrow{LTI(discr.)} |H(e^{j\omega})|\cos(\omega n + \measuredangle H(e^{j\omega}))$$

Standard first and second order low-pass systems, continuous time

$$H(j\omega) = \frac{1}{1 + j\omega\tau} \qquad H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

Standard first and second order recursive systems, discrete time

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \qquad |a| < 1$$

$$H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}} \qquad 0 \leq r < 1,\ 0 \leq \theta \leq \pi$$

Continuous time sampling

$$x_p(t) = x(t) \times p(t) \qquad x_d[n] = x(nT)$$

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \qquad \omega_s = \frac{2\pi}{T}$$

$$X_d(e^{j\omega}) = X_p(j\omega f_s) = X_p(j\omega/T)$$

$$H_0(j\omega) = e^{-j\omega T/2}\sin(\omega T/2)/(\omega/2) \quad \text{(sample and hold)}$$

Other formulas

$$A\cos(\phi + \theta) = A\sin(\phi + \theta + \pi/2) = B\cos\phi - C\sin\phi$$

$$A^2 = \sqrt{B^2 + C^2} \qquad \theta = \tan^{-1}\left(\frac{C}{B}\right)$$

$$ae^{j\phi} + a^* e^{-j\phi} = 2\,\mathrm{Re}\{ae^{j\phi}\} = 2|a|\cos(\phi + \measuredangle a)$$

$$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

$$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1) + c$$

$$\int x\cos ax\, dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a} + c$$

$$\int x\sin ax\, dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} + c$$

$$\frac{d\,\mathrm{atan}(x)}{dx} = \frac{d\tan^{-1}(x)}{dx} = \frac{1}{1+x^2}$$

Properties – Continuous time Fourier series (C.T.F.S.)

Definitions:	$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \qquad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $a_0 = \frac{1}{T}\int_T x(t)dt$ $x(t) \xleftrightarrow{C.T.F.S.} a_k \qquad y(t) \xleftrightarrow{C.T.F.S.} b_k$ $x(t)$ periodic with period T sec., Fundam. angular frequency $\omega_0 = 2\pi f_0 = 2\pi/T$ rad./sec.
Linearity:	$Ax(t)+By(t) \xleftrightarrow{C.T.F.S.} A\,a_k + B\,b_k$
Shifting:	$x(t-t_0) \xleftrightarrow{C.T.F.S.} e^{-jk\omega_0 t_0} a_k$
Scaling:	$x(\alpha t) \xleftrightarrow{C.T.F.S.} a_k$ ($\alpha > 0$, period T/α)
Flipping:	$x(-t) \xleftrightarrow{C.T.F.S.} a_{-k}$
Conjugate:	$x^*(t) \xleftrightarrow{C.T.F.S.} a_{-k}^*$ $x^*(-t) \xleftrightarrow{C.T.F.S.} a_k^*$
Symmetries:	if $x(t)$ is real: $a_k = a_{-k}^*$, $\lvert a_k \rvert = \lvert a_{-k} \rvert$, $\angle a_k = -\angle a_{-k}$ $x(t)$ real and even : a_k real and even $a_k = a_{-k}$ $x(t)$ real and odd: a_k imaginary and odd $a_k = -a_{-k}$
Periodic convolution:	$\int_T x(\tau)y(t-\tau)d\tau \xleftrightarrow{C.T.F.S.} T\,a_k\,b_k$
Modulation:	$x(t)y(t) \xleftrightarrow{C.T.F.S.} a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$ $e^{jm\omega_0 t}x(t) \xleftrightarrow{C.T.F.S.} a_{k-m}$
Differenciation:	$\frac{dx(t)}{dt} \xleftrightarrow{C.T.F.S.} jk\omega_0 a_k$
Integration:	$\int_{\tau=-\infty}^{t} x(\tau)d\tau \xleftrightarrow{C.T.F.S.} \frac{a_k}{jk\omega_0}$ (if $a_0 = 0$)
Parseval:	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k \rvert^2$

Table of continuous time Fourier series (C.T.F.S.)

$x(t)$ periodic, period $T = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$ sec.	Fourier series coefficients a_k
$e^{j\omega_0 t}$	$a_1 = 1$ $a_k = 0$ elsewhere
$\cos(\omega_0 t)$	$a_1, a_{-1} = 1/2$ $a_k = 0$ elsewhere
$\sin(\omega_0 t)$	$a_1, a_{-1} = 1/(2j)$ $a_k = 0$ elsewhere
$\begin{cases} 1 & \lvert t \rvert < T_1 \\ 0 & T_1 < \lvert t \rvert < T/2 \end{cases}$ (periodic T)	$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \quad k \neq 0$ $a_k = \frac{2T_1}{T} = \frac{T_1\omega_0}{\pi} \quad k = 0$
1	$a_0 = 1$ $a_k = 0$ elsewhere
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$a_k = \frac{1}{T}$

Properties – Discrete time Fourier series (D.T.F.S.)

Definitions: $a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-j(k\frac{2\pi}{N})n}$ $\qquad x[n] = \sum_{k=<N>} a_k e^{j(k\frac{2\pi}{N})n}$ $a_0 = \frac{1}{N}\sum_{n=<N>} x[n]$ $x[n] \xleftrightarrow{D.T.F.S.} a_k \quad y[n] \xleftrightarrow{D.T.F.S.} b_k$ $x[n]$ periodic with period N samples (fundamental angular frequency $\omega_0 = \frac{2\pi}{N}$ rad./sample)
Periodicity: $x[n] \xleftrightarrow{D.T.F.S.} a_k = a_{k+N}$
Linearity: $Ax[n] + By[n] \xleftrightarrow{D.T.F.S.} Aa_k + Bb_k$
Shifting: $x[n-n_0] \xleftrightarrow{D.T.F.S.} e^{-jk\frac{2\pi}{N}n_0} a_k$
Flipping: $x[-n] \xleftrightarrow{D.T.F.S.} a_{-k}$
Conjugate: $x^*[n] \xleftrightarrow{D.T.F.S.} a_{-k}^*$ $x^*[-n] \xleftrightarrow{D.T.F.S.} a_k^*$
Symmetries: if $x[n]$ is real : $a_k = a_{-k}^*$, $\lvert a_k\rvert = \lvert a_{-k}\rvert$, $\angle a_k = -\angle a_{-k}$ $x[n]$ real and even : a_k real and even $a_k = a_{-k}$ $x[n]$ real and odd: a_k imaginary and odd $a_k = -a_{-k}$
Periodic convolution: $\sum_{m=<N>} x[m]y[n-m] \xleftrightarrow{D.T.F.S.} N\, a_k\, b_k$
Modulation: $x[n]y[n] \xleftrightarrow{D.T.F.S.} \sum_{l=<N>} a_l b_{k-l}$ $e^{jm\frac{2\pi}{N}n} x[n] \xleftrightarrow{D.T.F.S.} a_{k-m}$
Accumulation : $\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{D.T.F.S.} \frac{1}{\left(1-e^{-jk\frac{2\pi}{N}}\right)} a_k$ (if $a_0 = 0$)
Parseval: $\frac{1}{N}\sum_{n=<N>} \lvert x[n]\rvert^2 = \sum_{k=<N>} \lvert a_k\rvert^2$
Duality : if $x[n] \xleftrightarrow{DTFS} a_k$ then $a[n] \xleftrightarrow{DTFS} \frac{1}{N} x_{-k}$

Table of discrete time Fourier series (D.T.F.S.)

$x[n]$ periodic, period N samples	Fourier series coefficients a_k (periodic with period N)
$e^{j\omega_0 n}$	If $x[n]$ periodic with $\omega_0 = \frac{2\pi m}{N}$: $a_k = 1 \quad k = m, m \pm N, m \pm 2N, \ldots$ $a_k = 0$ elsewhere
$\cos(\omega_0 n)$	If $x[n]$ periodic with $\omega_0 = \frac{2\pi m}{N}$: $a_k = 1/2$ $k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots$ $a_k = 0$ elsewhere
$\sin(\omega_0 n)$	If $x[n]$ periodic with $\omega_0 = \frac{2\pi m}{N}$: $a_k = 1/(2j)$ $k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots$ $a_k = 0$ elsewhere
$\begin{cases} 1 & \lvert n\rvert \le N_1 \\ 0 & N_1 < \lvert n\rvert \le N/2 \end{cases}$ (periodic N, N even)	$a_k = \frac{\sin\left(\frac{2\pi}{N}k(N_1+1/2)\right)}{N\sin(\frac{\pi}{N}k)}$ $k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = (2N_1+1)/N \;\; k = 0, \pm N, \pm 2N, \ldots$
1	$a_k = 1 \quad k = 0, \pm N, \pm 2N, \ldots$ $a_k = 0$ elsewhere
$\sum_{m=-\infty}^{\infty} \delta[n-mN]$	$a_k = \frac{1}{N}$

Properties – Continuous time Fourier transform (C.T.F.T.)

Definitions:

$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt \qquad x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega$

$x(t)\xleftrightarrow{CTFT}X(j\omega) \qquad y(t)\xleftrightarrow{CTFT}Y(j\omega)$

ω in rad./sec.

$X(j\omega)=\sum_{k=-\infty}^{+\infty}2\pi\ a_k\delta(\omega-k\omega_0)$ if $x(t)$ periodic

Linearity: $ax(t)+by(t)\xleftrightarrow{CTFT}aX(j\omega)+bY(j\omega)$

Shifting: $x(t-t_0)\xleftrightarrow{CTFT}e^{-j\omega t_0}X(j\omega)$

Scaling: $x(at)\xleftrightarrow{CTFT}\frac{1}{|a|}X(\frac{j\omega}{a})$

Flipping: $x(-t)\xleftrightarrow{CTFT}X(-j\omega)$

Conjugate: $x^*(t)\xleftrightarrow{CTFT}X^*(-j\omega)$

$x^*(-t)\xleftrightarrow{CTFT}X^*(j\omega)$

Symmetries:

if $x(t)$ is real : $X(j\omega)=X^*(-j\omega)$,

$|X(j\omega)|=|X(-j\omega)|,\ \angle X(j\omega)=-\angle X(-j\omega)$

$x(t)$ real and even : $X(j\omega)$ real and even $X(j\omega)=X(-j\omega)$

$x(t)$ real and odd: $X(j\omega)$ imag., odd $X(j\omega)=-X(-j\omega)$

Convolution:

$x(t)*y(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau \overset{CTFT}{\leftrightarrow} X(j\omega)Y(j\omega)$

Modulation:

$x(t)y(t)\overset{CTFT}{\leftrightarrow}\frac{1}{2\pi}X(j\omega)*Y(j\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\theta)Y(j(\omega-\theta))d\theta$

$e^{j\omega_0 t}x(t)\xleftrightarrow{CTFT}X(j(\omega-\omega_0))$

$\cos(\omega_0 t)x(t)\xleftrightarrow{CTFT}\frac{1}{2}X(j(\omega-\omega_0))+\frac{1}{2}X(j(\omega+\omega_0))$

Differenciation: $\frac{dx(t)}{dt}\xleftrightarrow{CTFT}j\omega\ X(j\omega)$

Integration: $\int_{-\infty}^{t}x(\tau)d\tau\xleftrightarrow{CTFT}\frac{1}{j\omega}X(j\omega)+\pi X(j0)\delta(\omega)$

Differenciation in freq.: $tx(t)\xleftrightarrow{CTFT}j\frac{dX(j\omega)}{d\omega}$

Integration in freq.:

$-\frac{1}{jt}x(t)+\pi x(0)\delta(t)\xleftrightarrow{CTFT}\int_{-\infty}^{\omega}X(j\eta)d\eta$

Parseval: $\int_{-\infty}^{+\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$

Duality : if $x(t)\xleftrightarrow{CTFT}X(j\omega)$ then

$X(t)\xleftrightarrow{CTFT}2\pi\ x(-j\omega)$

Table of continuous time Fourier transforms (C.T.F.T.)

signal $x(t)$ typ. aperiodic	$X(j\omega)$ (ω in rad./sec.)
if $x(t)$ is periodic, with period $T=\frac{2\pi}{\omega_0}=\frac{1}{f_0}$ sec.	$2\pi\sum_{k=-\infty}^{+\infty}a_k\delta(\omega-k\omega_0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
$\sin(\omega_0 t)$	$\frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)$
$\begin{cases}1 & \|t\|<T_1\\ 0 & T_1<\|t\|<T/2\end{cases}$ (periodic T)	$2\sum_{k=-\infty}^{+\infty}\frac{\sin(k\omega_0 T_1)}{k}\delta(\omega-k\omega_0)$ $k\neq 0$ $\frac{4\pi T_1}{T}\delta(\omega)=2T_1\omega_0\delta(\omega)$ $k=0$
1	$2\pi\delta(\omega)$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_s\sum_{k=-\infty}^{+\infty}\delta(\omega-k\omega_s)\quad \omega_s=\frac{2\pi}{T}$
$\begin{cases}1 & \|t\|<T_1\\ 0 & \|t\|>T_1\end{cases}$	$\frac{2\sin(\omega T_1)}{\omega}$
$\frac{\sin(Wt)}{\pi t}\quad W>0$	$\begin{cases}1 & \|\omega\|\le W\\ 0 & \|\omega\|>W\end{cases}$
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$
$e^{-at}u(t)\qquad \mathrm{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
$-e^{-at}u(-t)\qquad \mathrm{Re}\{a\}<0$	$\frac{1}{a+j\omega}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)\quad \mathrm{Re}\{a\}>0$	$\frac{1}{(a+j\omega)^n}$
$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$ $\mathrm{Re}\{a\}<0$	$\frac{1}{(a+j\omega)^n}$
$e^{-at}\sin(\omega_0 t)u(t)$ $a>0\quad \omega_0\ge 0\quad a,\omega_0$ real	$\frac{\omega_0}{(j\omega+a)^2+\omega_0^2}$
$e^{-at}\cos(\omega_0 t)u(t)$ $a>0\quad \omega_0\ge 0\quad a,\omega_0$ real	$\frac{j\omega+a}{(j\omega+a)^2+\omega_0^2}$
$-e^{-at}\sin(\omega_0 t)u(-t)$ $a<0\quad \omega_0\ge 0\quad a,\omega_0$ real	$\frac{\omega_0}{(j\omega+a)^2+\omega_0^2}$
$-e^{-at}\cos(\omega_0 t)u(-t)$ $a<0\quad \omega_0\ge 0\quad a,\omega_0$ real	$\frac{j\omega+a}{(j\omega+a)^2+\omega_0^2}$

Properties – Discrete time Fourier transform (D.T.F.T.)

Definitions: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ $x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n}d\omega$ $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ $y[n] \xleftrightarrow{DTFT} Y(e^{j\omega})$ ω in rad./sample $X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\frac{2\pi}{N})$ if $x[n]$ periodic
Periodicity: $x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$
Linearity: $ax[n]+by[n] \xleftrightarrow{DTFT} aX(e^{j\omega})+bY(e^{j\omega})$
Shifting: $x[n-n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0}X(e^{j\omega})$ n_0 integer
Expansion, insertion of zeros: $x_{(k)}[n] \xleftrightarrow{DTFT} X(e^{jk\omega})$ where k is a positive integer $x_{(k)}[n] = x[n/k]$ if n is a multiple of k $x_{(k)}[n] = 0$ elsewhere
Flipping: $x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$
Conjugate: $x^*[n] \xleftrightarrow{DTFT} X^*(e^{-j\omega})$ $x^*[-n] \xleftrightarrow{DTFT} X^*(e^{j\omega})$
Symmetries: if $x[n]$ is real : $X(e^{j\omega}) = X^*(e^{-j\omega})$, $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$, $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ $x[n]$ real and even : $X(e^{j\omega})$ real, even $X(e^{j\omega}) = X(e^{-j\omega})$ $x[n]$ real, odd $X(e^{j\omega})$ imag., odd $X(e^{j\omega}) = -X(e^{-j\omega})$
Convolution: $x[n]*y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k] \overset{DTFT}{\leftrightarrow} X(e^{j\omega})Y(e^{j\omega})$
Modulation: $x[n]y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ $e^{j\omega_0 n}x[n] \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$
Accumulation: $\sum_{m=-\infty}^{n} x[m] \overset{DTFT}{\leftrightarrow} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{m=-\infty}^{+\infty} \delta(\omega - m2\pi)$
Differenciation in freq.: $nx[n] \xleftrightarrow{DTFT} j\frac{dX(e^{j\omega})}{d\omega}$
Parseval: $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$
Duality : If $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ then $X(t) \xleftrightarrow{CTFS} x_{-k}$

Table of discrete time Fourier transforms (D.T.F.T.)

signal $x[n]$ typ. aperiodic	$X(e^{j\omega})$ (periodic 2π, ω in rad./sample)
if $x[n]$ is periodic, with period N samples	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\frac{2\pi}{N})$
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - l2\pi)$
$\cos(\omega_0 n)$	$\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - l2\pi)$ $+\pi \sum_{l=-\infty}^{\infty} \delta(\omega + \omega_0 - l2\pi)$
$\sin(\omega_0 n)$	$\frac{\pi}{j} \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - l2\pi)$ $-\frac{\pi}{j} \sum_{l=-\infty}^{\infty} \delta(\omega + \omega_0 - l2\pi)$
$\begin{cases} 1 & \lvert n\rvert \le N_1 \\ 0 & N_1 < \lvert n\rvert \le N/2 \end{cases}$ (periodic N, N even)	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\frac{2\pi}{N})$
1	$2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - l2\pi)$
$\sum_{m=-\infty}^{\infty} \delta[n - mN]$	$\frac{2\pi}{N} \sum_{m=-\infty}^{+\infty} \delta(\omega - m\frac{2\pi}{N})$
$\begin{cases} 1 & \lvert n\rvert \le N_1 \\ 0 & \lvert n\rvert > N_1 \end{cases}$	$\sin(\omega(N_1 + 1/2)) / \sin(\omega/2)$
$\frac{\sin(Wn)}{\pi n}$ $0 < W < \pi$	$\begin{cases} 1 & 0 \le \lvert\omega\rvert \le W \\ 0 & W < \lvert\omega\rvert \le \pi \end{cases}$ period. 2π
$\delta[n]$	1
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$
$a^n u[n]$ $\lvert a\rvert < 1$	$1/(1 - ae^{-j\omega})$
$-a^n u[-n-1]$ $\lvert a\rvert > 1$	$1/(1 - ae^{-j\omega})$
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n]$ $\lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$
$r^n \sin(\omega_0 n)u[n]$ $0 \le r < 1$ $0 \le \omega_0 \le \pi$	$\frac{r\sin(\omega_0)e^{-j\omega}}{1 - 2r\cos(\omega_0)e^{-j\omega} + r^2 e^{-j2\omega}}$
$r^n \cos(\omega_0 n)u[n]$ $0 \le r < 1$ $0 \le \omega_0 \le \pi$	$\frac{1 - r\cos(\omega_0)e^{-j\omega}}{1 - 2r\cos(\omega_0)e^{-j\omega} + r^2 e^{-j2\omega}}$
$-r^n \sin(\omega_0 n)u[-n-1]$ $r > 1$ $0 \le \omega_0 \le \pi$	$\frac{r\sin(\omega_0)e^{-j\omega}}{1 - 2r\cos(\omega_0)e^{-j\omega} + r^2 e^{-j2\omega}}$
$-r^n \cos(\omega_0 n)u[-n-1]$ $r > 1$ $0 \le \omega_0 \le \pi$	$\frac{1 - r\cos(\omega_0)e^{-j\omega}}{1 - 2r\cos(\omega_0)e^{-j\omega} + r^2 e^{-j2\omega}}$

Properties – bilateral (two-sided) Laplace transform

Definitions: $X(s)=\int_{-\infty}^{+\infty}x(t)e^{-st}dt$ $x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds$ $x(t)\overset{LT}{\longleftrightarrow}X(s)$ ROC_x $y(t)\overset{LT}{\longleftrightarrow}Y(s)$ ROC_y
Linearity: $ax(t)+by(t)\overset{LT}{\longleftrightarrow}aX(s)+bY(s)$ $ROC_x\cap ROC_y$
Shifting: $x(t-t_0)\overset{LT}{\longleftrightarrow}e^{-st_0}X(s)$ ROC_x unchanged
Scaling: $x(at)\overset{LT}{\longleftrightarrow}\frac{1}{\lvert a\rvert}X(\frac{s}{a})$ ROC_x dilated factor $\lvert a\rvert$ or compressed factor $\frac{1}{\lvert a\rvert}$,   and ROC_x inversed if $a<0$ **Flipping:** $x(-t)\overset{LT}{\longleftrightarrow}X(-s)$ ROC_x inversed
Conjugate: $x^*(t)\overset{LT}{\longleftrightarrow}X^*(s^*)$ ROC_x unchanged
Symmetry: if $x(t)$ real : $X(s)=X^*(s^*)$, $\lvert X(s)\rvert=\lvert X(s^*)\rvert$
Convolution: $x(t)*y(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau\overset{LT}{\leftrightarrow}X(s)Y(s)$ $ROC_x\cap ROC_y$
Modulation: $e^{s_0t}x(t)\overset{LT}{\longleftrightarrow}X(s-s_0)$ ROC_x shifted to right by $\operatorname{Re}\{s_0\}$
Differenciation: $\frac{dx(t)}{dt}\overset{LT}{\longleftrightarrow}s\,X(s)$ ROC_x unchanged
Integration: $\int_{-\infty}^{t}x(\tau)d\tau\overset{LT}{\longleftrightarrow}\frac{1}{s}X(s)$ $ROC_x\cap(\operatorname{Re}\{s\}>0)$
Differenciation in freq.: $-tx(t)\overset{LT}{\longleftrightarrow}\frac{dX(s)}{ds}$ ROC_x unchanged

Table of bilateral (two-sided) Laplace transforms

Signal $x(t)$	Laplace transform $X(s)$	ROC
$\delta(t)$	1	$\forall s$
$u(t)$	$1/s$	$\operatorname{Re}\{s\}>0$
$-u(-t)$	$1/s$	$\operatorname{Re}\{s\}<0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\operatorname{Re}\{s\}>-a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\operatorname{Re}\{s\}<-a$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$\operatorname{Re}\{s\}>-a$
$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$\operatorname{Re}\{s\}<-a$
$e^{-at}\sin(\omega_0t)u(t)$ $\omega_0\geq 0$ a,ω_0 real	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\operatorname{Re}\{s\}>-a$
$e^{-at}\cos(\omega_0t)u(t)$ $\omega_0\geq 0$ a,ω_0 real	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\operatorname{Re}\{s\}>-a$
$-e^{-at}\sin(\omega_0t)u(-t)$ $\omega_0\geq 0$ a,ω_0 real	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\operatorname{Re}\{s\}<-a$
$-e^{-at}\cos(\omega_0t)u(-t)$ $\omega_0\geq 0$ a,ω_0 real	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\operatorname{Re}\{s\}<-a$