MULTIPLE-CHOICE QUESTIONS

Questions 1–5 are multiple choice format worth 2 points each. Answers to multiple-choice questions do not need to be justified. You may write your scrap work on your paper but it will not be graded. When you reach your answer, clearly indicate the question number and write the letter of your response beside the question number: For example: *(write out your scrap work, but it will not be graded)*

(clearly indicate your final choice) **Q1.** [letter of your choice]

Q1. Use **logarithmic differentiation** to find the derivative of the function: $f(x) = (\ln(x))^{\ln(x)}$

What then is $f'(e^2)$?

Solution: **G**

A. 0 **B.** $2\ln(2)$ **C.** $4\ln(2)$ **D.** e^2+1

E. 1 **F.** $e^2\ln(2)$ **G.** $\dfrac{4+4\ln(2)}{e^2}$ **H.** $\dfrac{1+\ln(2)}{e^2}$

Q2. Based on a Riemann sum with n rectangles and right endpoints, which of the following limits defines the definite integral below?

$$\int_5^7 \left(x^2-4\right)\,dx$$

Solution: **E**

A. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4i^2}{n^2}-4\right)\left(\frac{2}{n}\right)$ **B.** $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(5+\frac{12i}{n}+\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$ **C.** $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(12+\frac{16i}{n}+\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$

D. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^2}{n^2}-4\right)\left(\frac{1}{n}\right)$ **E.** $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(21+\frac{20i}{n}+\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$ **F.** $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(9+\frac{24i}{n}+\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$

Q3. Find $\displaystyle\frac{d}{dt}\left[\int_{4t}^{5}\frac{4x-2}{x^2+20}\,dx\right]$.

Solution: **D**

A. $\dfrac{2-16t}{16t^2+20}$ **B.** $\dfrac{64t-8}{16t^2+20}$ **C.** $\dfrac{4t-2}{t^2+20}$ **D.** $\dfrac{8-64t}{16t^2+20}$

E. $\dfrac{16t-2}{16t^2+20}$ **F.** $\dfrac{2-4t}{t^2+20}$ **G.** $\dfrac{16t-8}{t^2+20}$ **H.** $\dfrac{8-16t}{t^2+20}$

Q4. Consider the integrable piecewise function defined as follows: $f(x) = \begin{cases} 5x^2 & \text{if } x \le 4 \\ 5x+60 & \text{if } x > 4 \end{cases}$

Find the value of the definite integral $\displaystyle\int_3^7 f(x)\,dx$. Solution: **B**

A. $\frac{559}{6}$ **B.** $\frac{1945}{6}$ **C.** $\frac{1399}{6}$ **D.** $\frac{727}{6}$ **E.** $\frac{881}{6}$ **F.** $\frac{1135}{6}$

Q5. Use the substitution $u = x^2$ to transform the following integral: $\int x^9 \tan(x^8)\, dx$.

What is the new integral in terms of u ? Solution: **A**

A. $\frac{1}{2}\int u^4 \tan(u^4)\, du$ **B.** $\frac{1}{2}\int u^5 \tan(u^4)\, du$ **C.** $\frac{1}{9}\int u^4 \tan(u^5)\, du$

D. $\frac{1}{9}\int u^4 \tan(u^4)\, du$ **E.** $\frac{1}{9}\int u^5 \tan(u^4)\, du$ **F.** $\frac{1}{2}\int u^4 \tan(u^5)\, du$

LONG-ANSWER QUESTIONS

For long-answer questions, all of your work must be justified and your steps must be written in a clear and logical order, using correct mathematical notation. Clearly indicate Question numbers.

For example: **Q7 a).** [write a fully justified solution].

Q6. [3 points] An animated film shows a circle whose radius and area vary with time (but its form always remains a circle). At a certain moment in time,

- the circles's area is $\frac{\pi}{25}$ cm^2 and its area is decreasing 9 cm^2/s.

At this moment in time, what is the rate of change of the circles's radius?

Give a fully justified solution. Clearly define any variables that appear in your solution and draw a diagram. Give your answer using appropriate units.

Solution: Let A denote the circle's area, let R denote its radius. We want to find $\frac{dR}{dt}$ at the moment in time when

$$A = \frac{\pi}{25}, \quad \frac{dA}{dt} = -9.$$

Since $A = \pi R^2$, at this moment, $R = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{\frac{\pi}{25}}{\pi}} = \frac{1}{5}$ cm. We have:

$$A = \pi R^2$$

$$\Rightarrow \frac{dA}{dt} = 2\pi R \frac{dR}{dt}$$

$$\Rightarrow -9 = 2\pi \left(\frac{1}{5}\right) \frac{dR}{dt}$$

$$\Rightarrow \frac{dR}{dt} = -\frac{5(9)}{2\pi} = -\frac{45}{2\pi} \text{ cm/s}.$$

Q7. [4 points] Find the unique function $g(x)$ that satisfies $g'(1) = 4$ and $g(1) = 2$, and

$$g''(x) = \frac{5}{x^2} + e^{1-x}.$$

Show your work!

Solution:

$$g''(x) = \frac{5}{x^2} + e^{1-x}$$

$$\Rightarrow g'(x) = -\frac{5}{x} - e^{1-x} + C$$

$$\Rightarrow 4 = -\frac{5}{a} - e^{0} + C$$

$$\Rightarrow C = 4 + 5 + 1 = 10$$

$$\Rightarrow g'(x) = -\frac{5}{x} - e^{1-x} + 10$$

$$\Rightarrow g(x) = -5\ln|x| + e^{1-x} + 10x + D$$

$$\Rightarrow 2 = -5\ln|1| + e^{0} + 10(1) + D$$

$$\Rightarrow D = 2 - 1 - 10 = -9$$

$$\Rightarrow g(x) = -5\ln|x| + e^{1-x} + 10x + -9$$

Q8. [8 points] Evaluate each of the following integrals. You must show ALL your steps and write your solutions in a logical order, using appropriate mathematical notation throughout.

a) $\displaystyle\int_e^{e^4} \frac{1}{x(\ln(x)+1)^4}\,dx$ b) $\displaystyle\int 1t\sin(4t)\,dt$ c) $\displaystyle\int \left(\frac{1+x^2}{\sqrt{x}} + \frac{1}{1+x^2}\right)dx$

Solution: a)

$$\begin{aligned}\int_e^{e^4} \frac{1}{x(\ln(x)+1)^4}\,dx &= \int_2^5 \frac{1}{xu^4}(x\,du) && \text{sub: } u = \ln(x)+1 \Rightarrow du = \tfrac{1}{x}\,dx \Rightarrow dx = x\,du \\ &= \int_2^5 u^{-4}\,du && x = e^4 \Rightarrow u = \ln(e^4)+1 = 4+1 = 5 \\ &= \left[-\frac{1}{3}u^{-3}\right]_2^5 && x = e \Rightarrow u = \ln(e)+1 = 1+1 = 2 \\ &= \left(-\frac{1}{3}(5)^{-3}\right) - \left(-\frac{1}{3}(2)^{-3}\right)\end{aligned}$$

Solution: b)

$$\int 1t\sin(4t)\,dt = 1t\left(-\frac{1}{4}\cos(4t)\right) - \int 1\left(-\frac{1}{4}\cos(4t)\right)\,dt \qquad \text{parts: } u = 1t \quad v' = \sin(4t)$$

$$= -\frac{1t}{4}\cos(4t) + \frac{1}{4}\int \cos(4t)\,dt \qquad u' = 1 \quad v = -\tfrac{1}{4}\cos(4t)$$

$$= -\frac{1t}{4}\cos(4t) + \frac{1}{4}\left(\frac{1}{4}\sin(4t)\right) + C$$

$$= -\frac{1t}{4}\cos(4t) + \frac{1}{16}\sin(4t) + C$$

Solution: c)

$$\int \left(\frac{1+x^2}{\sqrt{x}} + \frac{1}{1+x^2}\right)\,dx = \int \left(\frac{1}{\sqrt{x}} + \frac{x^2}{\sqrt{x}} + \frac{1}{1+x^2}\right)\,dx$$

$$= \int \left(x^{-1/2} + x^{3/2} + \frac{1}{1+x^2}\right)\,dx$$

$$= 2x^{1/2} + \frac{2}{5}x^{5/2} + \arctan(x) + C$$

To compute $\int_0^1 e^{3x}\cos(x-1)dx$, let $t=x-1$ it becomes
$\int_{-1}^0 e^{3(t+1)}\cos(t)dt = e^3\int_{-1}^0 e^{3t}\cos(t)dt = e^3 I = \frac{1}{10}[3e^3+\sin(1)-3\cos(1)]$

$$\begin{aligned}
I &= \int_{-1}^0 e^{3t}\cos(t)dt = \int_{-1}^0 e^{3t}d\sin(t) = \left[e^{3t}sin(t)\right]_{-1}^0 - \int_{-1}^0 \sin(t)3e^{3t}dt \\
&= \left[e^{3t}sin(t)\right]_{-1}^0 - 3\int_{-1}^0 e^{3t}(-\cos(t))'dt = \left[e^{3t}\sin(t)\right]_{-1}^0 - 3\left[e^{3t}(-\cos(t)) - \int_{-1}^0 3e^{3t}(-\cos(t))dt\right] \\
&= \left[e^{3t}\sin(t)+3e^{3t}\cos(t)\right]_{-1}^0 - 9\int_{-1}^0 e^{3t}\cos(t)dt \\
10I &= \left[e^{3t}\sin(t)+3e^{3t}\cos(t)\right]_{-1}^0 \\
I &= \frac{1}{10}\left[e^{3t}\sin(t)+3e^{3t}\cos(t)\right]_{-1}^0 = \frac{1}{10}[e^0\sin(0)+3e^0\cos(0)-(e^{-3}\sin(-1)+3e^{-3}\cos(-1))] \\
&= \frac{1}{10}[3+e^{-3}\sin(1)-3e^{-3}\cos(1)]
\end{aligned}$$

$$\left[e^{3t}\sin(t)+3e^{3t}\cos(t)\right]' = 3e^{3t}\sin(t)+e^{3t}\cos(t)+9e^{3t}\cos(t)-3e^{3t}\sin(t) = 10e^{3t}\cos(t)$$

End of the Exam!