MULTIPLE-CHOICE QUESTIONS

Your answers to multiple-choice questions do not need to be justified. You may write your scrap work on your paper but it will not be graded. When you reach your answer, clearly indicate the question number and write the letter of your response beside the question number: For example: (*write out your scrap work, but it will not be graded)*

(*clearly indicate your final choice*) **Q1.** [letter of your choice]

Q1. The dimensions of a cylinder are changing over time, but at all times, its surface area satisfies the equation below:

$$A = 2\pi R^2 + 2\pi RH.$$

At some moment in time, the cylinder's radius is $R = 10$ cm and its radius is decreasing at a rate of 1 cm/min, and the cylinder's height is $H = 5$ cm and its height is increasing at a rate of 3 cm/min.

At this moment, what is the rate of change of this cylinder's surface area A ?

Solution: D

A. A is decreasing by 10π cm^2/min
B. A is decreasing by 20π cm^2/min
C. A is decreasing by 30π cm^2/min
D. A is increasing by 10π cm^2/min
E. A is increasing by 20π cm^2/min
F. A is increasing by 30π cm^2/min

Q2. Based on a Riemann sum with n rectangles and right endpoints, which of the following limits defines the definite integral below?

$$\int_2^5 (2x+1)\,dx$$

Solution: D

A. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}+1\right)\left(\frac{3}{n}\right)$
B. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{6i}{n}+6\right)\left(\frac{3}{n}\right)$
C. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{6i}{n}+1\right)\left(\frac{3}{n}\right)$
D. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{6i}{n}+5\right)\left(\frac{3}{n}\right)$
E. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n}+7\right)\left(\frac{3}{n}\right)$
F. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n}+4\right)\left(\frac{3}{n}\right)$

Q3. Find $\frac{d}{dx}\left[\int_3^{\sin(5x)} e^{(t^2)}\,dt\right]$.

Solution: C

A. $e^{(x^2)} + C$
B. $-5\sin(5x)e^{\cos^2(5x)}$
C. $5\cos(5x)e^{\sin^2(5x)}$
D. $e^{\sin^2(5x)}$
E. $e^{(x^2)}$
F. $5\cos(5x)e^{(x^2)}$

Q4. Suppose you know $\int_3^1 f(x)\,dx = -9$ and $\int_3^5 f(x)\,dx = -6$. Find $\int_1^5 f(x)\,dx$.

Solution: **F**

A. -15 **B.** 6 **C.** -6 **D.** -3 **E.** 9 **F.** 3 **G.** -9 **H.** 15

Q5. For the integral below, if we use integration by parts with the part $f(x) = \ln(x)$ and the appropriate part $g'(x)$, what do we get for $g(x)$?

$$\int \frac{\ln(x)}{\sqrt{x}}\,dx$$

Solution: **F**

A. $g(x) = -2x^{-1/2}$ **B.** $g(x) = \frac{2}{3}x^{3/2}$ **C.** $g(x) = \frac{3}{2}x^{3/2}$

D. $g(x) = \frac{1}{2}x^{-1/2}$ **E.** $g(x) = -\frac{2}{3}x^{-3/2}$ **F.** $g(x) = 2x^{1/2}$

Q6. Choose an appropriate u-substitution and transform the definite integral given below:

$$\int_e^{e^3} \frac{dx}{x(\ln(x))^4}$$

In terms of u, which of the following is equal to the above integral?

Solution: **D**

A. $\displaystyle\int_e^{e^3} u^{-4}\,du$ **B.** $\displaystyle -\frac{1}{4}\int_e^{e^3} u^{-4}\,du$ **C.** $\displaystyle -\tfrac{1}{4}\int_1^{3} u^{-4}\,du$

D. $\displaystyle\int_1^{3} u^{-4}\,du$ **E.** $\displaystyle \tfrac{1}{x}\int_e^{e^3} u^{-4}\,du$ **F.** $\displaystyle \tfrac{1}{x}\int_1^{3} u^{-4}\,du$

LONG-ANSWER QUESTIONS

For long-answer questions, all of your work must be justified and your steps must be written in a clear and logical order. Clearly indicate Question numbers.

For example: **Q7 a).** [write a fully justified solution].

Q7. a) Without simplifying your answer, differentiate $f(x) = x\arctan\left(\frac{1}{x}\right) + \frac{1}{2}\ln(x^2+1)$.

Solution:

$$f(x) = x\arctan\left(\tfrac{1}{x}\right) + \tfrac{1}{2}\ln(x^2+1)$$

$$\Rightarrow f'(x) = (1)\arctan\left(\tfrac{1}{x}\right) + x\left(\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-x^{-2}\right)\right) + \frac{1}{2}\left(\frac{1}{x^2+1}(2x)\right)$$

b) Now, simplify your answer from part (a) to show that $f'(x) = \arctan\left(\frac{1}{x}\right)$.
Show your steps!

Solution:

$$\begin{aligned} f'(x) &= (1)\arctan\left(\frac{1}{x}\right) + x\left(\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-x^{-2}\right)\right) + \frac{1}{2}\left(\frac{1}{x^2+1}(2x)\right) \\ &= \arctan\left(\tfrac{1}{x}\right) - x\left(\frac{1}{1+\frac{1}{x^2}}\right)\left(\frac{1}{x^2}\right) + \frac{x}{x^2+1} \\ &= \arctan\left(\tfrac{1}{x}\right) - x\left(\frac{1}{x^2\left(1+\frac{1}{x^2}\right)}\right) + \frac{x}{x^2+1} \\ &= \arctan\left(\tfrac{1}{x}\right) - \frac{x}{x^2+1} + \frac{x}{x^2+1} \\ &= \arctan\left(\tfrac{1}{x}\right) \end{aligned}$$

c) Knowing from (b) that $\frac{d}{dx}\left[x\arctan\left(\frac{1}{x}\right) + \frac{1}{2}\ln(x^2+1)\right] = \arctan\left(\frac{1}{x}\right)$, use this fact to help you evaluate the following definite integral. Show your work and write an exact expression for the final answer as well as a numerical value, correct to 3 decimal places.

$$\int_1^{\sqrt{3}} \left(\arctan\left(\tfrac{1}{x}\right) + 7x^3\right)\,dx$$

Solution:

$$\begin{aligned} \int_1^{\sqrt{3}} \left(\arctan\left(\tfrac{1}{x}\right) + 7x^3\right)\,dx &= \left[x\arctan\left(\tfrac{1}{x}\right) + \tfrac{1}{2}\ln(x^2+1) + \tfrac{7}{4}x^4\right]_1^{\sqrt{3}} \\ &= \left(\sqrt{3}\arctan\left(\tfrac{1}{\sqrt{3}}\right) + \tfrac{1}{2}\ln((\sqrt{3})^2+1) + \tfrac{7}{4}(\sqrt{3})^4\right) \\ &\quad - \left((1)\arctan\left(\tfrac{1}{1}\right) + \tfrac{1}{2}\ln(1^2+1) + \tfrac{7}{4}(1)^4\right) \\ &= \left(\sqrt{3}\left(\tfrac{\pi}{6}\right) + \tfrac{1}{2}\ln(4) + \tfrac{9(7)}{4}\right) - \left(\left(\tfrac{\pi}{4}\right) + \tfrac{1}{2}\ln(2) + \tfrac{7}{4}\right) \\ &\approx 14.468 \end{aligned}$$

Q8. Evaluate each of the following integrals. You must show ALL your steps and write your solutions in a logical order, using appropriate mathematical notation throughout.

a) $\displaystyle\int \frac{2x^4-3x+2}{x^2}\,dx$ b) $\displaystyle\int \sec^2(\theta)\tan^7(\theta)\,d\theta.$ c) $\displaystyle\int_1^{e^2} \sqrt{t}\ln(t^5)\,dt$

Solution: a)

$$\begin{aligned} \int \frac{2x^4-3x+2}{x^2}\,dx &= \int \left(2x^2 - \frac{3}{x} + 2x^{-2}\right)dx \\ &= \frac{2}{3}x^3 - 3\ln|x| - 2x^{-1} + C \end{aligned}$$

Solution: b) We use the substitution $u = \tan(\theta)$. We find $\frac{du}{d\theta} = \sec^2(\theta)$. We then obtain:

$$\begin{aligned}
\int \sec^2\theta \tan^7\theta \, d\theta &= \int \sec^2\theta u^7 \cdot \frac{du}{\sec^2\theta} \\
&= \int u^7 \, du \\
&= \frac{1}{8}u^8 + C \\
&= \frac{1}{8}\tan^8(\theta) + C.
\end{aligned}$$

Solution: c) We first simplify the integrand:

$$\int_1^{e^2} \sqrt{t}\ln(t^5)\, dt = 5\int_1^{e^2} \sqrt{t}\ln(t)\, dt$$

We then use integration by parts with $u = \ln(t)$ and $v' = \sqrt{t} = t^{1/2}$.

Thus, $u' = \frac{1}{t}$ and $v = \frac{2}{3}t^{3/2}$.

$$\begin{aligned}
\int_1^{e^2} \sqrt{t}\ln(t^5)\, dt &= 5\int_1^{e^2} \sqrt{t}\ln(t)\, dt \\
&= 5\left[\ln(t)\left(\frac{2}{3}t^{3/2}\right) - \int \left(\frac{1}{t}\right)\left(\frac{2}{3}t^{3/2}\right) dt\right]_1^{e^2} \\
&= 5\left[\ln(t)\left(\frac{2}{3}t^{3/2}\right) - \frac{2}{3}\int t^{1/2}\, dt\right]_1^{e^2} \\
&= 5\left[\ln(t)\left(\frac{2}{3}t^{3/2}\right) - \frac{2}{3}\left(\frac{2}{3}t^{3/2}\right)\right]_1^{e^2} \\
&= \left[\frac{10}{3}t^{3/2}\ln(t) - \frac{20}{9}t^{3/2}\right]_1^{e^2} \\
&= \left(\frac{10}{3}(e^2)^{3/2}\ln(e^2) - \frac{20}{9}(e^2)^{3/2}\right) - \left(\frac{10}{3}(1)^{3/2}\ln(1) - \frac{20}{9}(1)^{3/2}\right) \\
&= \frac{10}{3}e^3(2) - \frac{20}{9}e^3 - 0 + \frac{20}{9} \\
&= \frac{40e^3 + 20}{9} \\
&= 91.4912\ldots \\
&\approx 91.491
\end{aligned}$$