

## Solution to final Examination

MATY1330, Fall 2022

### Part I. Multiple-Choice Questions ( $2 \times 6 = 12$ points)

1. The population  $p_t$  of honey bees after  $t$  weeks in a hive fluctuates according to the linear discrete-time dynamic system

$$p_{t+1} = -0.25p_t + 1500.$$

Which one of the following is a general solution to this system when  $p_0 = 600$ ?

- (A)  $p_t = 600 \times (-0.25)^t + 1200$ ;                      (B)  $p_t = -600 \times (-0.25)^t + 1200$ ;  
(C)  $p_t = -600 \times 0.25^t + 1200$ ;                      (D)  $p_t = 1200 \times (-0.25)^t + 600$ ;  
(E)  $p_t = -1400 \times (-0.25)^t + 2000$ ;                      (F)  $p_t = -1400 \times 0.25^t + 2000$ .

*Solution.* (B) We know the general solution has the form  $p_t = ar^t + b$ , where  $r = -0.25$ .

$$p_0 = a + b = 600, \quad p_1 = (-0.25)a + b = -0.25 \times 600 + 1500 = 1350.$$

$$\text{Hence, } 1.25a = -750. \quad a = -600. \quad b = 600 - a = 600 + 600 = 1200.$$

The general solution is  $p_t = -600 \times (-0.25)^t + 1200$ .

*Alternative Solution.* Use the formula. The general solution of a linear DTDS  $m_{i+1} = rm_i + k$  is  $m_i = (m_0 - x^*)r^i + x^*$ , where  $x^*$  is the equilibrium  $x^* = \frac{k}{1-r}$ . In this question  $r = -0.25$ ,  $k = 1500$ ,  $x^* = \frac{1500}{1.25} = 1200$ . Then  $m_0 - x^* = -600$ . Hence, the general solution is  $p_t = -600 \times (-0.25)^t + 1200$ .

2. For which value(s) of  $a$  is the following function continuous for all real numbers  $x$ ?

$$f(x) = \begin{cases} e^{2x} + 3, & x < -2 \\ a|x-1|, & x \geq -2 \end{cases}$$

- (A)  $a = \frac{e^4}{3} + 1$ ;                      (B)  $a = \frac{e^{-4}}{3} + 1$ ;                      (C)  $a = e^{-4} + 3$ ;  
(D)  $a = -\frac{e^{-4}}{3} - 1$ ;                      (E)  $a = e^4 + 3$ ;                      (F) None of the above.

*Solution.* (B) Let  $\lim_{x \rightarrow (-2)^-} f(x) = e^{-4} + 3 = \lim_{x \rightarrow (-2)^-} a|x-1| = a|-2-1| = 3a$ ,  $a = \frac{e^{-4} + 3}{3} = \frac{1}{3}e^{-4} + 1$ .

3. Which one of the following is the derivative of the function  $h(t) = \ln\left(\frac{3^t}{\sqrt{t^2+1}}\right)$ ?

(A)  $h'(t) = \frac{3}{t} - \frac{t}{t^2+1}$ ;      (B)  $h'(t) = \frac{\sqrt{t^2+1}}{3^t}$ ;      (C)  $h'(t) = \ln 3 - \frac{t^2}{2(t^2+1)}$ ;

(D)  $h'(t) = \ln 3 + \frac{t}{t^2+1}$ ;      (E)  $h'(t) = \ln 3 - \frac{t}{t^2+1}$ ;      (F)  $h'(t) = \frac{3}{t} - \frac{1}{t^2+1}$ .

*Solution.* (E) By the chain rule,  $h'(t) = \frac{\sqrt{t^2+1}}{3^t} \left( \frac{d}{dt} \left( \frac{3^t}{\sqrt{t^2+1}} \right) \right)$ .

By the quotient rule,  $\frac{d}{dt} \left( \frac{3^t}{\sqrt{t^2+1}} \right) = \frac{(\ln 3)3^t \sqrt{t^2+1} - 3^t \frac{t}{\sqrt{t^2+1}}}{t^2+1} = \frac{3^t(\ln 3(t^2+1) - t)}{(t^2+1)\sqrt{t^2+1}}$ .

Hence,  $h'(t) = \frac{\sqrt{t^2+1}}{3^t} \left( \frac{3^t(\ln 3(t^2+1) - t)}{(t^2+1)\sqrt{t^2+1}} \right) = \frac{\ln 3(t^2+1) - t}{t^2+1} = \ln 3 - \frac{t}{t^2+1}$ .

4. The second derivative of the function  $g(x) = \sin(x^2)$  is

(A)  $-\sin(x^2)$ ;      (B)  $-2x \sin(x) + 2 \cos x$ ;      (C)  $-4x^2 \sin(x^2) + 2 \cos(x^2)$ ;  
 (D)  $4x^2 \sin(x^2) + 2 \cos(x^2)$ ;      (E)  $-4x \sin(x^2)$ ;      (F)  $2x \sin(x) + 2 \cos(x)$ .

*Solution.* (C)  $g'(x) = 2x \cos(x^2)$ .  $g''(x) = 2 \cos(x^2) + 2x(-\sin(x^2))(2x) = 2 \cos(x^2) - 4x^2 \sin(x^2)$ .

5. Let  $f(x)$  be a differentiable function defined on the closed interval  $[2, 3]$ . Assume that  $f(2) = 7$  and  $f(3) = 3$ . Which two of the following statements are true? (Choose two of the following!)

(A) There must exist a value  $c$  in interval  $[2, 3]$  such that  $f(c) = 0$ .

(B) There must exist a value  $c$  in interval  $[2, 3]$  such that  $f(c) = 5$ .

(C) There must exist a value  $c$  in interval  $[2, 3]$  such that  $f(c) = -4$ .

(D) There must exist a value  $c$  in interval  $[2, 3]$  such that  $f'(c) = 0$ .

(E) There must exist a value  $c$  in interval  $[2, 3]$  such that  $f'(c) = 5$ .

(F) There must exist a value  $c$  in interval  $[2, 3]$  such that  $f'(c) = -4$ .

*Answer.* (B) and (F). Since  $3 < 5 < 7$ , there must be a value  $c$  in interval  $[2, 3]$  such that  $f(c) = 5$ .

Since the slope of the secant line joining  $(2, 7)$  and  $(3, 3)$  is  $m = \frac{3-7}{3-2} = -4$ , there must be a value  $c$  in interval  $[2, 3]$  such that  $f'(c) = -4$ .

6. Let  $f(x) = x^5 - \frac{1}{3}x^4 + 2x^3 + x^2 + 8x - 3$ . Which one of the following is Taylor's polynomial of degree 3, i.e.,  $T_3(x)$ , of  $f(x)$  at  $x = 0$ ?

(A)  $\frac{2}{3!}x^3 + \frac{1}{2!}x^2 + 8x - 3$ ;

(B)  $x^3 - \frac{1}{3}x^2 + 2x + 1$ ;

(C)  $\frac{1}{3!}x^3 - \frac{1}{3 \times 2!}x^2 + 2x + 1$ ;

(D)  $2x^3 + x^2 + 8x - 3$ ;

(E)  $x^5 - \frac{1}{3}x^4 + 2x^3 + x^2$ ;

(F)  $12x^3 + 2x^2 + 8x - 3$ .

*Solution.* (D)  $f(0) = -3$ .  $f'(x) = 5x^4 - \frac{4}{3}x^3 + 6x^2 + 2x + 8$ .  $f'(0) = 8$ ,  $f''(x) = 20x^3 - 4x^2 + 12x + 2$ ,  $f''(0) = 2$ .  $f^{(3)}(x) = 60x^2 - 8x + 12$ ,  $f^{(3)}(0) = 12$ .

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 = -3 + 8x + x^2 + 2x^3.$$

## Part II. Short-Answer Questions ( $2 \times 4 = 8$ points)

**In this part, only the final answer will be marked.**

7. Find the equation of the tangent line of the graph of the equation  $xe^y + y = x^2$  at the point  $(1, 0)$ .

*Answer.* The equation of the tangent line is

$$y = (1/2)x - 1/2.$$

*Solution.* Take the derivative with respect to  $x$  on both sides of this equation.  $e^y + xe^y y' + y' = 2x$ . When  $x = 1$  and  $y = 0$ ,  $1 + y' + y' = 2$ .  $y' = 1/2$ .

The equation of the tangent line is  $y = (1/2)(x - 1) + 0$ , or  $y = (1/2)x - 1/2$ .

**8.** The cost (in dollars) to produce quality hammers depends on the purity of steel used. Let the purity index of the steel be  $p$  and let the cost of producing 100 hammers with steel of quality index  $p$  be  $C(p)$ . Study shows that

$$C(p) = 2p^3 + 9p^2 - 60p + 300, 0 \leq p \leq 5.$$

What is the maximum possible cost and what is the minimum possible production cost of producing 100 hammers?

**Answer.** The maximum cost of producing 100 quality hammers is \$475.

The minimum cost of producing 100 quality hammers is \$232.

*Solution.* Let  $C' = 6p^2 + 18p - 60 = 6(p^2 + 3p - 10) = 0$ .  $p = 2$ . Since  $C''(2) > 0$ ,  $C(p)$  attains a local minimum at  $p = 2$ .  $C(2) = 232$ . Since  $C(0) = 300$  and  $C(5) = 475$ . The global maximum is 475, and the global minimum is 232.

**9.** Consider equation  $x^3 - x - 1 = 0$ . Use Newton's method to find an approximation of a root of this equation starting with initial approximation  $x_0 = 2$ . Do two iterations. Use six digits after the decimal point in your calculation and round the final result to five decimal places.

**Answer.**  $x_2 = \underline{1.35962}$ .

*Solution.* Let  $f(x) = x^3 - x - 1$ . Then  $f'(x) = 3x^2 - 1$ .

The iteration formula is  $x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$ .

$$x_0 = 2,$$

$$x_1 = 2 - 2 - \frac{2^3 - 2 - 1}{3 \times 2^2 - 1} = 2 - \frac{5}{11} \approx 1.545454,$$

$$x_2 = 1.545454 - \frac{1.545454^3 - 1.545454 - 1}{3 \times 1.545454^2 - 1} \approx 1.359615.$$

**10.** What is the **exact** value of the definite integral  $I = \int_1^3 \left( x^2 - 4 + \frac{1}{x} \right) dx$ ?

**Answer.**  $I = \frac{2}{3} + \ln 3$ .

$$\text{Solution. } \int_1^3 \left( x^2 - 4 + \frac{1}{x} \right) dx = \left[ \frac{1}{3} x^3 - 4x + \ln x \right]_{x=1}^3 = \frac{26}{3} - 8 + \ln 3 = \frac{2}{3} + \ln 3.$$

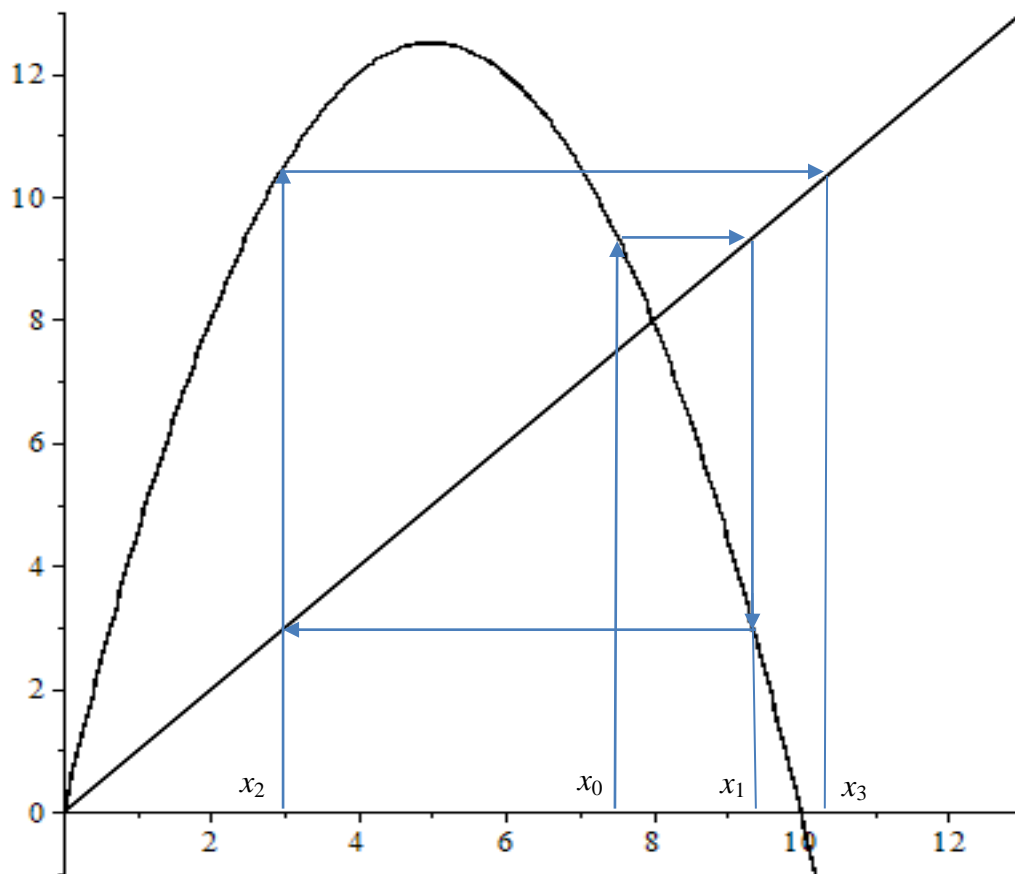
**Part III. Detailed-Answer Questions** (30 points)

**11.** (10 points) A farmer noticed that the amount of hay produced by one of their untended fields is modeled by the discrete-time dynamic system with recursive relation

$$b_{t+1} = \frac{1}{2} b_t(10 - b_t),$$

where  $t$  is measured in years, and  $b_t$  is the number of bales in 100's. This system has equilibria  $b_1^* = 0$  and  $b_2^* = 8$ .

(a) (2 points) The graph of the updating function of this system with the line  $y = x$  is given in the following figure:



If  $b_0 = 7.5$ . Use cobwebbing in the figure above to locate  $b_1$ ,  $b_2$ , and  $b_3$  on the horizontal axis.

(b) (1 point) According your result in part (a), is the equilibrium  $b_2^* = 8$  stable or unstable?

*Answer.* This equilibrium is unstable.

(c) The farmer decides to start tending this hay field. In particular, they introduced a *harvesting rate*  $h > 0$  in this discrete-time dynamic system so that the recursive relation of the new system becomes

$$b_{t+1} = \frac{1}{2} b_t(10 - b_t) - hb_t,$$

where  $t$  is in years and  $b_t$  is in 100's of bales.

(i) (1 point) Write the updating function of this system.

*Answer.*  $f(x) = \frac{1}{2}x(10 - x) - hx.$

(ii) (2 points) Find all equilibria of this system. Equilibria may depend on the parameter  $h$ .

*Solution.* Let  $x = \frac{1}{2}x(10 - x) - hx$ . Then  $x = 0$  or  $10 - x - 2h = 2$ ,  $x = 8 - 2h$ . This system has two equilibria  $b_1^* = 0$ , and  $b_2^* = 8 - 2h$ .

(iii) (1 point) For which value  $h$ , does this system have a positive equilibrium?

*Solution.* Let  $8 - 2h > 0$ . Then  $h < 4$ . When  $h < 4$ , this system has a positive equilibrium  $b_2^* = 8 - 2h > 0$ .

(iv) (3 points) Use the Stability Theorem to determine the range of  $h$  in which the positive equilibrium is stable.

*Solution.* Find the derivative of the updating function:

Write  $f(x) = -\frac{1}{2}x^2 + (5 - h)x$ .

$$f'(x) = -x + (5 - h).$$

At the positive equilibrium,  $x = 8 - 2h$ ,  $h < 4$ .

$$f'(8 - 2h) = 2h - 8 + 5 - h = h - 3.$$

According to the Stability Theorem, the equilibrium  $8 - 2h$  is stable if  $|h - 3| < 1$ , or  $-1 < h - 3 < 1$ . Then  $2 < h < 4$ .

**12.** (5 points) Compute (not guess!) the following limits:

(a) (2 points)  $\lim_{x \rightarrow 0^+} (x^2 + x) \ln x$ .

*Solution.*  $\lim_{x \rightarrow 0^+} (x^2 + x) \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/(x^2 + x)} = \lim_{x \rightarrow 0^+} \frac{1/x}{-(2x+1)/(x^2 + x)^2} = - \lim_{x \rightarrow 0^+} \frac{(x^2 + x)^2}{x(2x+1)}$   
 $= - \lim_{x \rightarrow 0^+} \frac{x(x+1)^2}{2x+1} = 0$ .

(b) (3 points)  $\lim_{x \rightarrow (\pi/2)^-} (\tan x)^{1/\ln(\frac{\pi}{2}-x)}$ .

Recall that  $\frac{d}{dx} \tan x = \sec^2 x = \frac{1}{\cos^2 x}$ .

*Solution.* Let  $y = (\tan x)^{1/\ln(\frac{\pi}{2}-x)}$ . Then  $\ln y = \frac{\ln(\tan x)}{\ln(\frac{\pi}{2}-x)}$ .

$$\lim_{x \rightarrow (\frac{\pi}{2})^+} \ln y = \lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{\ln(\tan x)}{\ln(\frac{\pi}{2}-x)} = \lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{(1/\tan x)(\sec^2 x)}{-\frac{1}{\pi/2-x}} = - \lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{\pi/2-x}{\tan x \cos^2 x} = - \lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{\pi/2-x}{\sin x \cos x}$$

$$= - \lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{-1}{\cos^2 x - \sin^2 x} = -1.$$

$$\lim_{x \rightarrow (\frac{\pi}{2})^+} (\tan x)^{1/\ln(\frac{\pi}{2}-x)} = e^{-1}.$$

**13.** (5 points) Suppose a function  $y = f(x)$  defined for all  $x \neq -2$  satisfies the following conditions:

- (a)  $f'(x) < 0$  when  $x < -4$ , or  $-2 < x < 1$ , or  $x > 3$ ;  
 $f'(x) = 0$  when  $x = -4$ , or  $x = 1$ , or  $x = 3$ ;  
 $f'(x) > 0$  when  $-4 < x < -2$ , or  $1 < x < 3$ ;  
 $f'(x)$  is not define when  $x = -2$ .

- (b)  $f''(x) < 0$  when  $x < -5$  or  $x > 2$ ;  
 $f''(x) = 0$  when  $x = -5$  or  $x = 2$ ;  
 $f''(x) > 0$  when  $-5 < x < -2$  or  $-2 < x < 2$ ;  
 $f''(x)$  is not defined when  $x = -2$ .

(c)  $\lim_{x \rightarrow \infty} f(x) = -\infty$ ,  $\lim_{x \rightarrow -\infty} f(x) = 3$ .

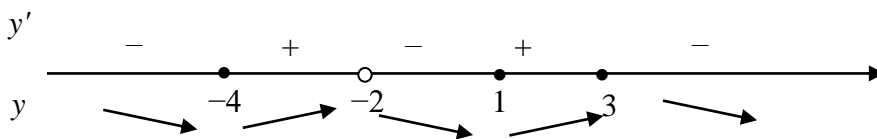
(d)  $\lim_{x \rightarrow (-2)^+} f(x) = \lim_{x \rightarrow (-2)^-} f(x) = \infty$ .

(e)  $f(-4) = 2, f(2) = 0$ .

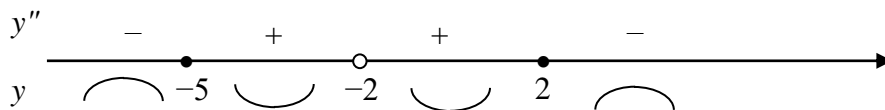
Sketch the graph of this function on the next page.

Your curve must show clearly the monotonicity and the concavity. Identify all local extrema, inflection point(s), and vertical / horizontal asymptote(s), if any.

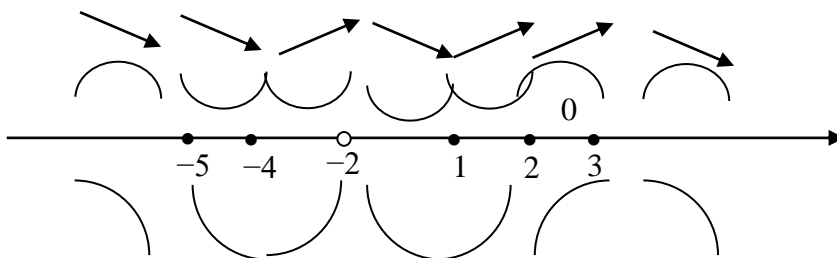
*Solution.* According to condition, we have the following diagram:



From (b), construct the following diagram:

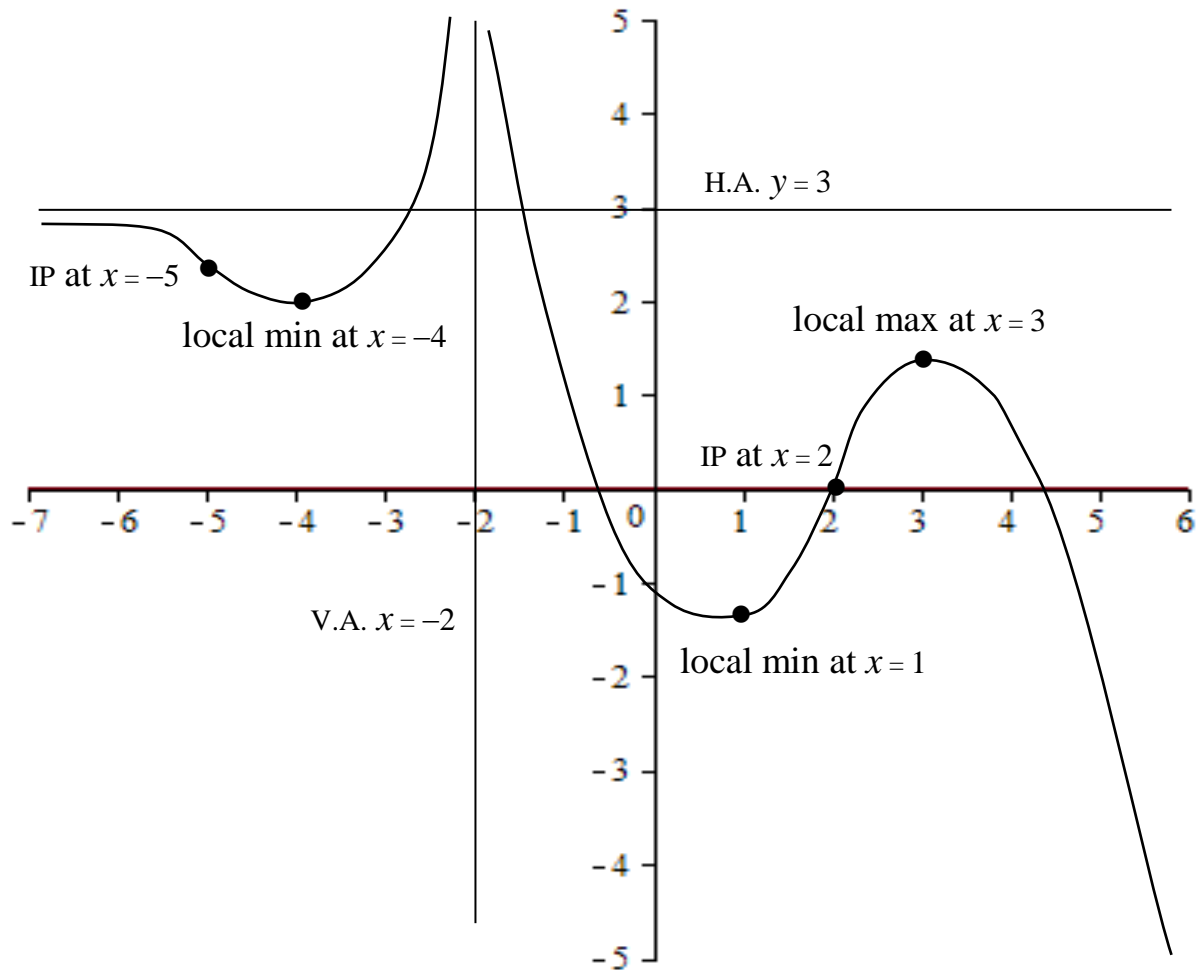


Combining the information obtained in these two diagrams, we have the shapes of sections of the graph:



Condition (c) and (d) gives horizontal asymptote  $y = 3$ , and vertical asymptote  $x = -2$ .

With special points  $(-4, 2)$  and  $(2, 0)$ , the graph of this function looks like the following:



14. (3 points) Find function  $y = f(x)$  if  $f''(x) = x^{-3/2}$ ,  $f(4) = -3$ , and  $f'(4) = 2$ .

*Solution.*  $f'(x) = -2x^{-1/2} + C$ .  $f'(4) = -1 + C = 2$ ,  $C = 3$ .

$f(x) = -4x^{1/2} + 3x + C$ .  $f(4) = -8 + 12 + C = -3$ ,  $C = -7$ .

$f(x) = -4x^{1/2} + 3x - 7$ .

15. (7 points) Calculate the following integrals:

(a) (2 points)  $\int \left( -x^3 + e^{-x} - \sin x + \frac{1}{\sqrt{1-x^2}} \right) dx =$

*Solution.*  $\int \left( -x^3 + e^{-x} - \sin x + \frac{1}{\sqrt{1-x^2}} \right) dx = -\frac{1}{4}x^4 - e^{-x} + \cos x + \arcsin x + C.$

(b) (2 points)  $\int_1^{e^\pi} \frac{\sin(\ln x)}{2x} dx =$

*Solution.* Let  $u = \ln x$ . Then  $u' = 1/x$ . When  $x = 1$ ,  $u = 0$ ; when  $x = e^\pi$ ,  $u = \pi$ .

$$\int_1^{e^\pi} \frac{\sin(\ln x)}{2x} dx = \int_0^\pi \sin u du = [-\cos x]_{x=0}^\pi = 2.$$

(c) (3 points)  $\int (x^2 - 3)e^{-x} dx.$

*Solution.* Use integration by parts. Let  $u = x^2 - 3$ , and let  $v' = e^{-x}$ . Then  $u' = 2x$  and  $v = -e^{-x}$ .

$$\int (x^2 - 3)e^{-x} dx = -(x^2 - 3)e^{-x} + 2 \int xe^{-x} dx.$$

Use integration by parts again. Let  $u = x$  and let  $v' = e^{-x}$ . Then  $u' = 1$ , and  $v = -e^{-x}$ .

$$\int xe^{-x} dx = -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} + C.$$

Hence,

$$\int (x^2 - 3)e^{-x} dx = -(x^2 - 3)e^{-x} - 2(xe^{-x} + e^{-x}) + C = -e^{-x}(x^2 + 2x - 1) + C.$$