1. The Jacobian of the map $\varphi : (x, y) \mapsto (u, v) = (y/x, xy)$ is

$$\operatorname{Jac}\varphi(x,y) = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix} = -\frac{2y}{x} \ .$$

The inverse map $\psi : (u, v) \mapsto (x, y)$ is $x = \sqrt{v/u}$, $y = \sqrt{uv}$. Therefore, φ and ψ establish a one-to-one correspondence between the sets $\{(x, y) : x, y > 0\}$ and $\{(u, v) : u, v > 0\}$. Thus, the joint density of U and V is

$$f_{UV}(u,v) = f_{XY}(x,y)|\operatorname{Jac}\varphi(x,y)|^{-1} = e^{-x-y}\cdot\frac{x}{2y} = \frac{1}{2u}e^{-\sqrt{v/u}-\sqrt{uv}}$$

for $u, v > 0$.

2. By putting $Y' = -Y$, so that Y' is uniformly distributed on $[0, 1]$, we can say that $Z = X + Y'$ [one can also work directly with X and Y]. Therefore, the density of Z is the convolution of the densities of X and of Y', i.e.,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_{Y'}(z-x)\, dx \ .$$

Since X and Y' are uniform, their densities take the constant values 1/2 and 1 on the respective intervals (0,2) and (0,1), whence

$$\begin{aligned} f_Z(z) &= \tfrac{1}{2}\cdot\operatorname{length}(\{x : 0 < x < 2, 0 < z - x < 1\}) \\ &= \tfrac{1}{2}\cdot\operatorname{length}((0,2)\cap(z-1,z)) \ , \end{aligned}$$

i.e.,

$$f_Z(z) = \begin{cases} 0\ , & z \le 0\ , \\ \frac{z}{2}\ , & 0 < z < 1\ , \\ \frac{1}{2}\ , & 1 \le z \le 2\ , \\ \frac{1}{2}(3-z)\ , & 2 < z < 3\ , \\ 0\ , & z \ge 3\ . \end{cases}$$

As for the expectation, $EZ = EX - EY = 1 + \frac{1}{2} = \frac{3}{2}$. [Alternatively, one can find EZ by using Y' or the density f_Z.]

3. The number X of representative triples is the sum of the indicators X_i of the events

$$A_i = \{i \text{ the triple centered at } i\text{-th position is representative}\}$$

with $1 \le i \le 90$. Then $P(A_i) = 3! \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{6}$ for all $1 \le i \le 90$, whence

$$EX = \sum_{i=1}^{90} EX_i = \sum_{i=1}^{90} P(A_i) = 90 \cdot \frac{1}{6} = 15\ .$$

4. Since the covariance and correlation for the couple (X, Y) are the same as for the couple $(X-2, Y-2)$, we may apply the operation of subtracting constant 2 and assume that the values of balls are $-1, 0, 1$ [one can also work directly with the initial ball values]. Then the distributions of both X and Y are uniform on the set of values $\{-1, 0, 1\}$, whence $EX = EY = 0$ and $EX^2 = EY^2 = \frac{2}{3}$, whence $\mathrm{Var}X = \mathrm{Var}Y = \frac{2}{3}$. Finally, $EXY = -\frac{1}{3}$ (there are 3 possible equiprobable pairs of balls resulting from drawing; for two of them that include the 0 value ball $XY = 0$, and $XY = -1$ for the remaining pair of balls with the values $-1, 1$). Thus, $\mathrm{Cov}XY = -\frac{1}{3}$ and $\mathrm{Corr}(X, Y) = -\frac{1}{2}$.

5. The joint density function f_{XY} splits into the product of the marginal density functions $f_X(x) = x/2$ for $x \in [0, 2]$ (and 0 otherwise), and $f_Y(y) = 2y$ for $y \in [0, 1]$ (and 0 otherwise), whence X and Y are independent, and therefore

$$\mathrm{Cov}(X + 2Y, X - 2Y) = \mathrm{Var}X - 4\mathrm{Var}Y \ ,$$

whereas

$$\mathrm{Var}(X + 2Y) = \mathrm{Var}(X - 2Y) = \mathrm{Var}X + 4\mathrm{Var}Y \ ,$$

with

$$\mathrm{Var}Y = EY^2 - (EY)^2 = 2\int_0^1 y^3\,dy - \left(2\int_0^1 y^2\,dy\right)^2 = \frac{1}{18} \ ,$$

and, in the same way,

$$\mathrm{Var}X = 4\,\mathrm{Var}Y = \frac{2}{9} \ ,$$

whence

$$\mathrm{Cov}(X + 2Y, X - 2Y) = 0$$

and

$$\mathrm{Corr}(X + 2Y, X - 2Y) = 0 \ .$$