Solution to Midterm Test 1

MAT1330, Fall 2022

Part I. Multiple-choice Questions ($2 \times 5 = 10$ points)

Version 1. CFBCA
Version 2. FEAEB
Version 3. BBDAC, or BDDAC
Version 4. ACCFD

1.1. Assume some values of functions $y = f(x)$ and $y = g(x)$ are given in the following table:

x	1	2	3	4	5	6
$f(x)$	2	4	5	3	6	1
$g(x)$	4	5	1	6	3	2

Then $(f \circ g)(1) =$

(A) 1; (B) 2; (C) 3; (D) 4; (E) 5; (F) 6.

Solution. (C) $(f \circ g)(1) = f(g(1)) = f(4) = 3$.

1.2. Assume some values of functions $y = f(x)$ and $y = g(x)$ are given in the following table:

x	1	2	3	4	5	6
$f(x)$	2	4	5	3	6	1
$g(x)$	4	5	1	6	3	2

Then $(f \circ g)(2) =$

(A) 1; (B) 2; (C) 3; (D) 4; (E) 5; (F) 6.

Solution. (F) $(f \circ g)(2) = f(g(2)) = f(5) = 6$.

1.3. Assume some values of functions $y = f(x)$ and $y = g(x)$ are given in the following table:

x	1	2	3	4	5	6
$f(x)$	2	4	5	3	6	1
$g(x)$	4	5	1	6	3	2

Then $(f \circ g)(3) =$

(A) 1; (B) 2; (C) 3; (D) 4; (E) 5; (F) 6.

Solution. (B) $(f \circ g)(3) = f(g(3)) = f(1) = 2$.

1.4. Assume some values of functions $y = f(x)$ and $y = g(x)$ are given in the following table:

x	1	2	3	4	5	6
$f(x)$	2	4	5	3	6	1
$g(x)$	4	5	1	6	3	2

Then $(f \circ g)(4) =$

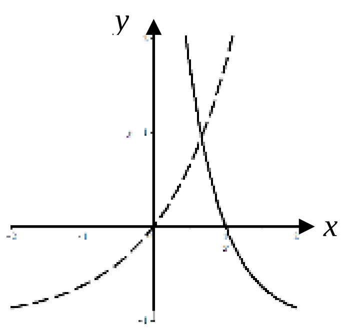
(A) 1; (B) 2; (C) 3; (D) 4; (E) 5; (F) 6.

Solution. (A) $(f \circ g)(4) = f(g(4)) = f(6) = 1$.

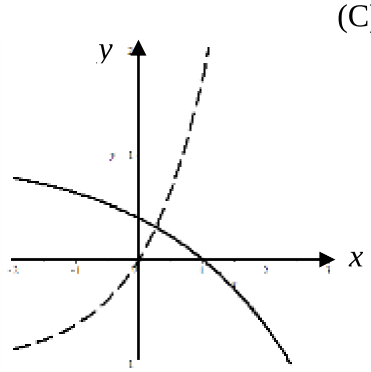
2.1. In the following figures, the dashed curve is the graph of a function $y = f(x)$. In which of the following figures, the other curve is the graph of function

$$y = -f\left(\frac{1}{2}(x+1)\right)?$$

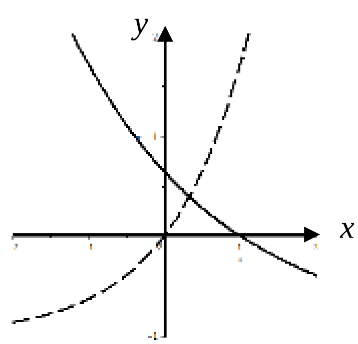
(A)

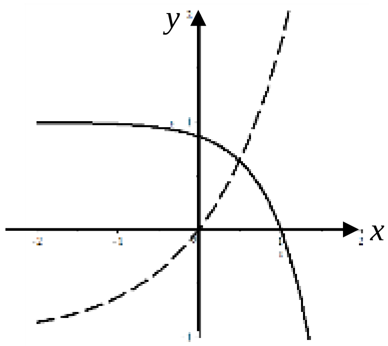
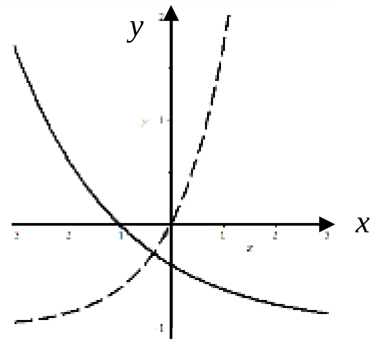
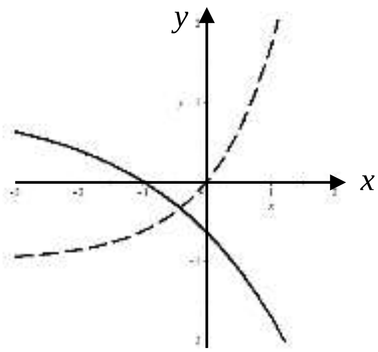


(B)

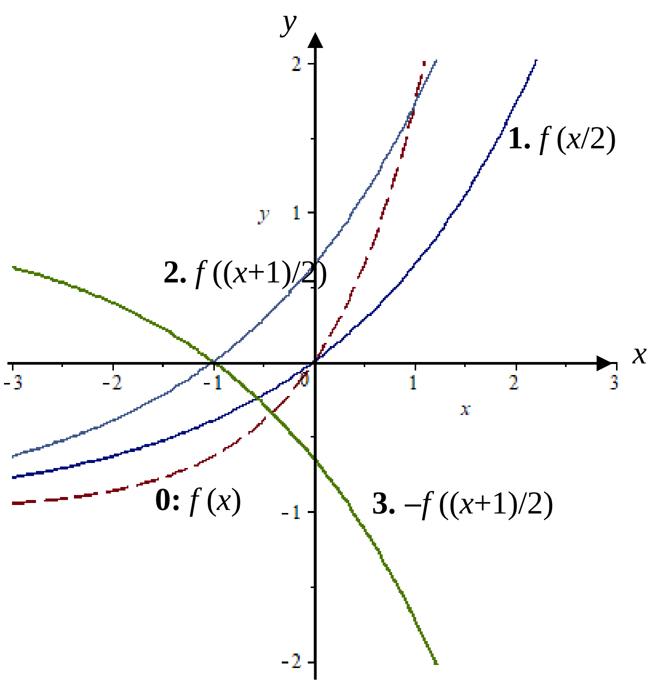


(C)



(D)  (E)  (F) 

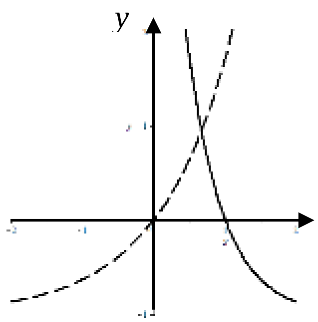
Solution. (F)



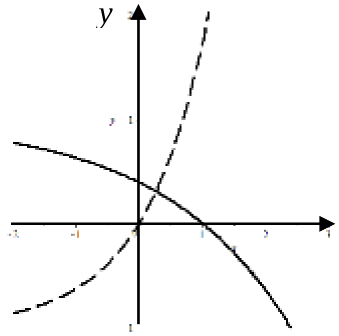
2.2. In the following figures, the dashed curve is the graph of a function $y = f(x)$. In which of the following figures, the other curve is the graph of function

$y = f\left(\frac{1}{2}(-x-1)\right)$?

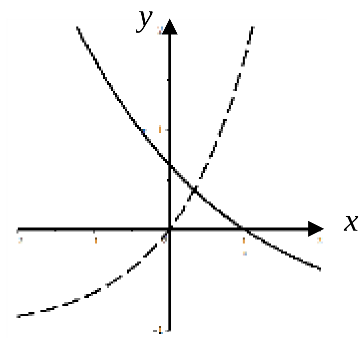
(A)



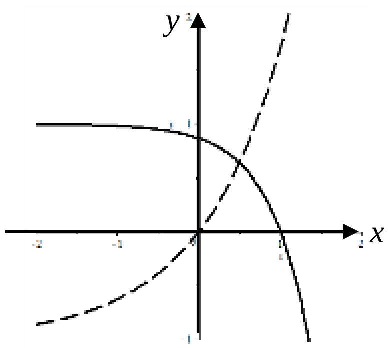
(B)



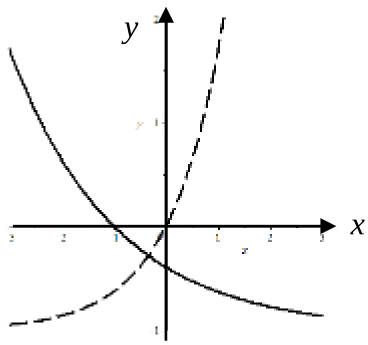
(C)



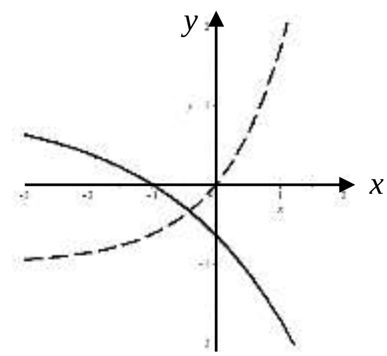
(D)



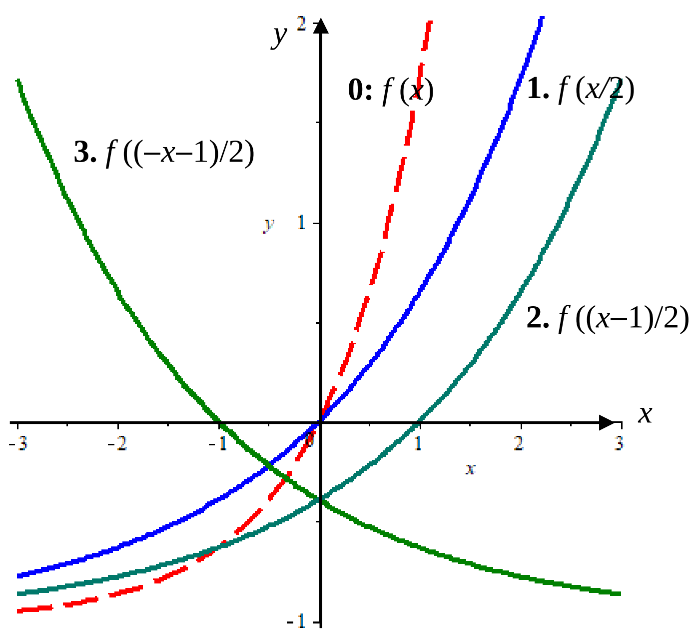
(E)



(F)



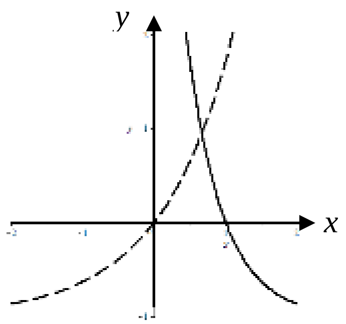
Solution. (E)



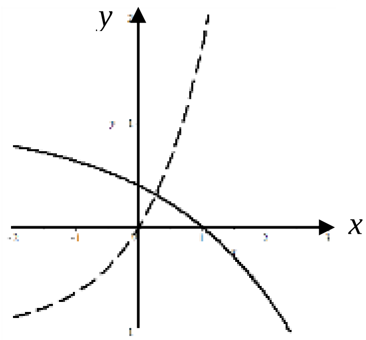
2.3. In the following figures, the dashed curve is the graph of a function $y = f(x)$. In which of the following figures, the other curve is the graph of function

$$y = -f\left(\frac{1}{2}(x-1)\right)?$$

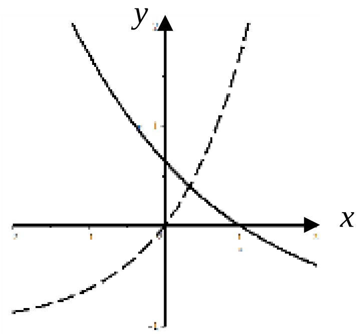
(A)



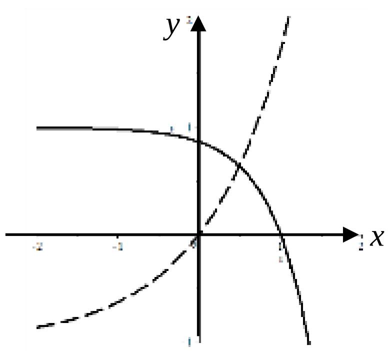
(B)



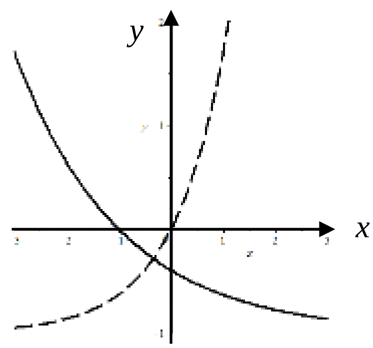
(C)



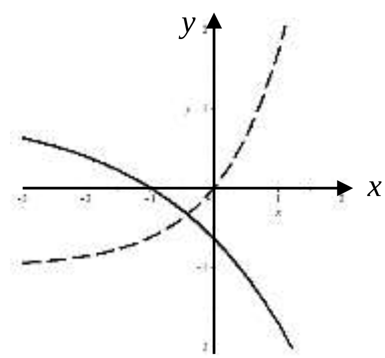
(D)



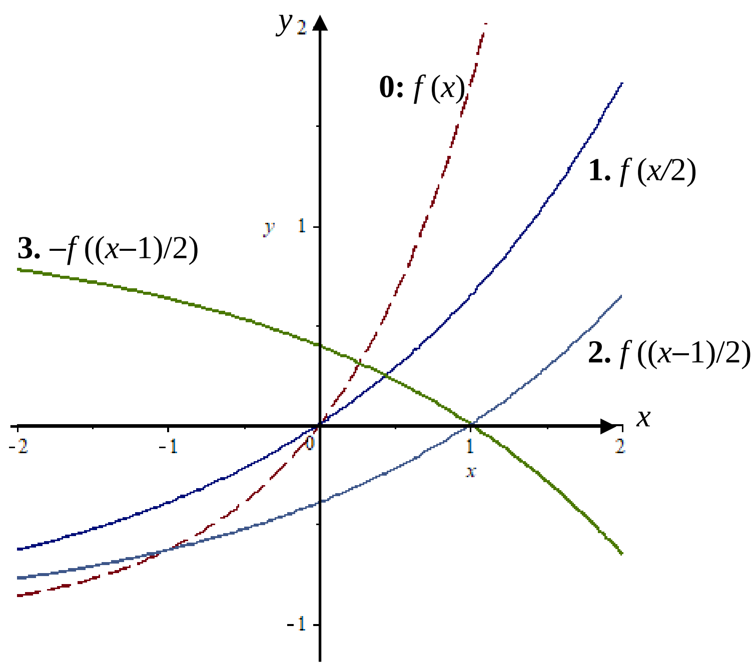
(E)



(F)



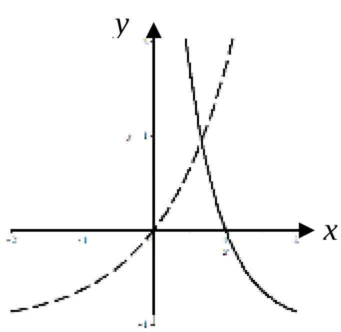
Solution. (B) is the correct answer. Because (D) is similar to (B), I would all accept (D).



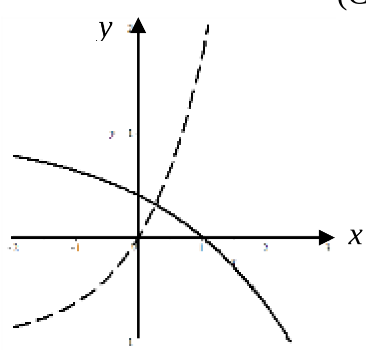
2.4. In the following figures, the dashed curve is the graph of a function $y = f(x)$. In which of the following figures, the other curve is the graph of function

$$y = f\left(\frac{1}{2}(-x+1)\right)?$$

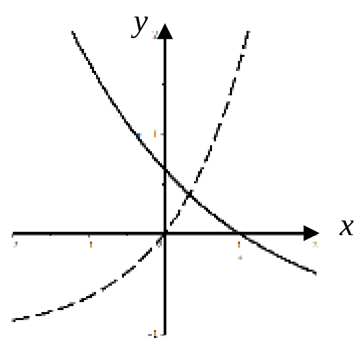
(A)



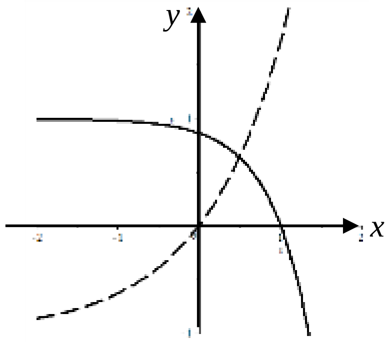
(B)



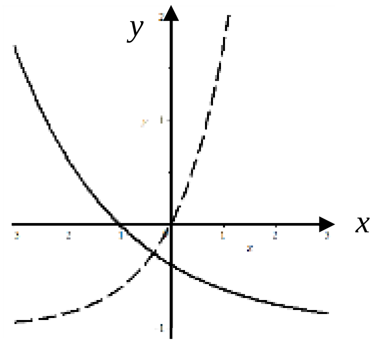
(C)



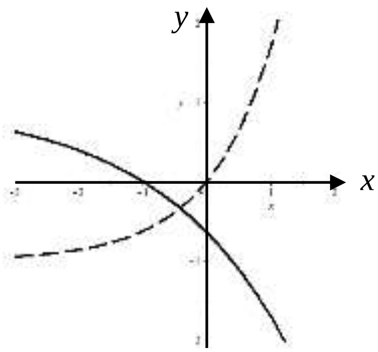
(D)



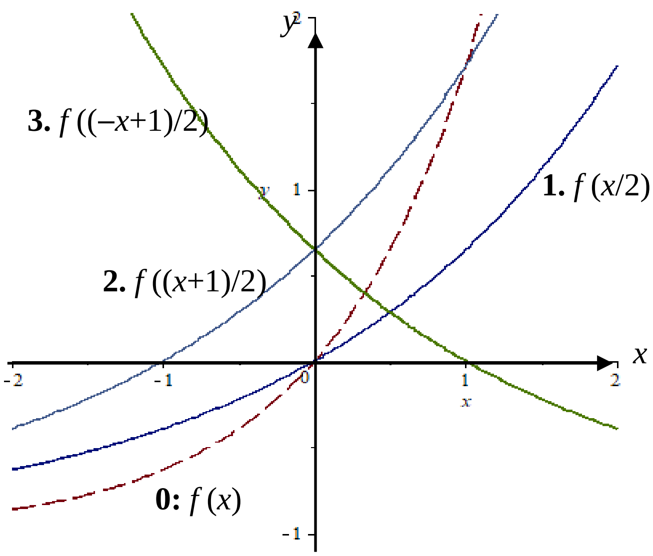
(E)



(F)



Solution. (C)



3.1. The equation of a line L going through point $(2, -1)$ and parallel to the line L_1 with the equation $3x - 2y = 1$ is

(A) $2x + 3y = 1$; (B) $3x - 2y = 8$; (C) $2x - 3y = 7$;
(D) $3x + 2y = 4$; (E) $2x + 3y = 7$; (F) $3x - 2y = 5$.

Solution. (B) $3x - 2y = 1$, $y = \frac{3}{2}x - \frac{1}{2}$. The slope of the line L_1 is $m_1 = \frac{3}{2}$. The slope of the line L is $m = m_1 = \frac{3}{2}$. The equation of L has the form $y = \frac{3}{2}x + b$. When $x = 2$, $y = -1$. $-1 = 3 + b$. $b = -4$. The equation of L is $y = \frac{3}{2}x - 4$, or $3x - 2y = 8$.

3.2. The equation of a line L going through point $(2, -1)$ and parallel to the line L_1 with the equation $2x + 3y = 6$ is

(A) $2x + 3y = 1$; (B) $3x - 2y = 8$; (C) $2x - 3y = 7$;
(D) $3x + 2y = 4$; (E) $2x + 3y = 7$; (F) $3x - 2y = 4$.

Solution. (A) $2x + 3y = 6$, $y = -\frac{2}{3}x + 2$. The slope of the line L_1 is $m_1 = -\frac{2}{3}$. The slope of the line L is $m = m_1 = -\frac{2}{3}$. The equation of L has the form $y = -\frac{2}{3}x + b$. When $x = 2$, $y = -1$. $-1 = -\frac{4}{3} + b$. $b = \frac{1}{3}$. The equation of L is $y = -\frac{2}{3}x + \frac{1}{3}$, or $2x + 3y = 1$.

3.3. The equation of a line L going through point $(2, -1)$ and parallel to the line L_1 with the equation $3x + 2y = 1$ is

(A) $2x + 3y = 1$; (B) $3x - 2y = 8$; (C) $2x - 3y = 7$;
(D) $3x + 2y = 4$; (E) $2x + 3y = 7$; (F) $3x - 2y = 4$.

Solution. (D) $3x + 2y = 1$, $y = -\frac{3}{2}x + \frac{1}{2}$. The slope of the line L_1 is $m_1 = -\frac{3}{2}$. The slope of the line L is $m = m_1 = -\frac{3}{2}$. The equation of L has the form $y = -\frac{3}{2}x + b$. When $x = 2$, $y = -1$. $-1 = -3 + b$. $b = 2$. The equation of L is $y = -\frac{3}{2}x + 2$, or $3x + 2y = 4$.

3.4. The equation of a line L going through point $(2, -1)$ and parallel to the line L_1 with the equation $2x - 3y = 1$ is

(A) $2x + 3y = 1$; (B) $3x - 2y = 8$; (C) $2x - 3y = 7$;
(D) $3x + 2y = 4$; (E) $2x + 3y = 7$; (F) $3x - 2y = 5$.

Solution. (C) $2x - 3y = 1$, $y = \frac{2}{3}x - \frac{1}{3}$. The slope of the line L_1 is $m_1 = \frac{2}{3}$. The slope of the line L is $m = m_1 = \frac{2}{3}$. The equation of L has the form $y = \frac{2}{3}x + b$. When $x = 2$, $y = -1$. $-1 = \frac{4}{3} + b$. $b = -\frac{7}{3}$. The equation of L is $y = \frac{2}{3}x + \frac{7}{3}$, or $2x - 3y = 7$.

4.1. Suppose the population of a kind of animal in a region satisfies the exponential model. If the population is 850 at the beginning of the year 2020, and the population becomes 1000 at the beginning of the year 2022, which one of the following numbers (in years) is closest to the doubling time of this model?

(A) 7.60; (B) 8.01; (C) 8.53; (D) 8.98; (E) 9.34; (F) 9.95.

Solution. (C) Let the population t years after the beginning of 2020 be $\underline{P}(t)$. The model is $P(t) = P(0)e^{rt} = 850e^{kt}$. Since $P(2) = 1000 = 850e^{2k}$, $e^{2k} = 1000 / 850 = 20 / 17$. $k = \ln (20 / 17) / 2$.
The doubling time is $T = \ln 2 / k = 2 \ln 2 / \ln (20 / 17) \approx 8.53$ years.

4.2. Suppose the number of people infected by a virus in a short period of time satisfies the exponential model. If the number of patients increases from 250 to 290 in two days, which one of the following numbers (in days) is closest to the doubling time of this model?

(A) 7.60; (B) 8.01; (C) 8.53; (D) 8.98; (E) 9.34; (F) 9.95.

Solution. (E) Let the number of patients at day t be $c(t)$. The model is $c(t) = c(0)e^{rt}$. Since $c(2) = 290 = 250e^{2k}$, $e^{2k} = 290 / 250 = 29 / 25 = 1.16$. $k = \ln 1.16 / 2$.
The doubling time is $T = \ln 2 / k = 2 \ln 2 / \ln 1.16 \approx 9.34$ days.

4.3. Suppose the mass of a piece of radioactive substance is reduced from 1.2 grams to 1 gram in 2 seconds. Which one of the following numbers (in seconds) is closest to the half-life of this radioactive substance?

(A) 7.60; (B) 8.01; (C) 8.53; (D) 8.98; (E) 9.34; (F) 9.95.

Solution. (A) Let the mass of this piece ot radioactive substance t years be $\underline{m}(t)$. The model is $m(t) = m(0)e^{rt} = 1.2e^{kt}$. Since $m(2) = 1 = 1.2e^{2k}$, $k = \ln (1 /1.2) / 2 = -\ln 1.2 / 2$.

The half-life is $T = -\ln 2 / k = -2 \ln 2 / (-\ln 1.2) \approx 7.60$ seconds.

4.4. Suppose the amount of a kind of medication in the body of a patient satisfies the exponential decay model. If the amount of the medication is reduced to 87% in 2 hours, which one of the following numbers (in hours) is closest to the half-life of this model?

(A) 7.60; (B) 8.01; (C) 8.53; (D) 8.98; (E) 9.34; (F) 9.95.

Solution. (F) Let the amount of medication t hours be $A(t)$. The model is $A(t) = A(0)e^{rt} = A(0)e^{kt}$.
Since $P(2) = 0.87A(0) = A(0)e^{2k}$, $e^{2k} = 1 / 0.87$. $k = \ln 0.87 / 2$.
The half-life is $T = -\ln 2 / k = -2 \ln 2 / \ln 0.87 \approx 9.95$ hours.

5.1. If function $f(x) = \begin{cases} 2 + ax & x \geq 1 \\ x^2 + 2a & x < 1 \end{cases}$ is continuous for all real numbers x, then a =

(A) 1; (B) 2; (C) 3; (D) −1; (E) −2; (F) −3.

Solution. (A) Let $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$. Then $2 + a = 1 + 2a$, $a = 1$.

5.2. If function $f(x) = \begin{cases} a - x & x \geq 2 \\ ax - x^2 & x < 2 \end{cases}$ is continuous for all real numbers x, then a =

(A) 1; (B) 2; (C) 3; (D) −1; (E) −2; (F) −3.

Solution. (B) Let $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$. Then $a - 2 = 2a - 4$, $a = 2$.

5.3. If function $f(x) = \begin{cases} 2 - ax & x \geq -1 \\ 2a - x^2 & x < -1 \end{cases}$ is continuous for all real numbers x, then a =

(A) 1; (B) 2; (C) 3; (D) −1; (E) −2; (F) −3.

Solution. (C) Let $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$. Then $2 + a = 2a - 1$, $a = 3$.

5.4. If function $f(x) = \begin{cases} 1 + 2ax & x \geq 1 \\ 2x^2 + 3a & x < 1 \end{cases}$ is continuous for all real numbers x, then a =

(A) 1; (B) 2; (C) 3; (D) −1; (E) −2; (F) −3.

Solution. (D) Let $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$. Then 1 + 2$a$ = 2 + 3a, a = −1.

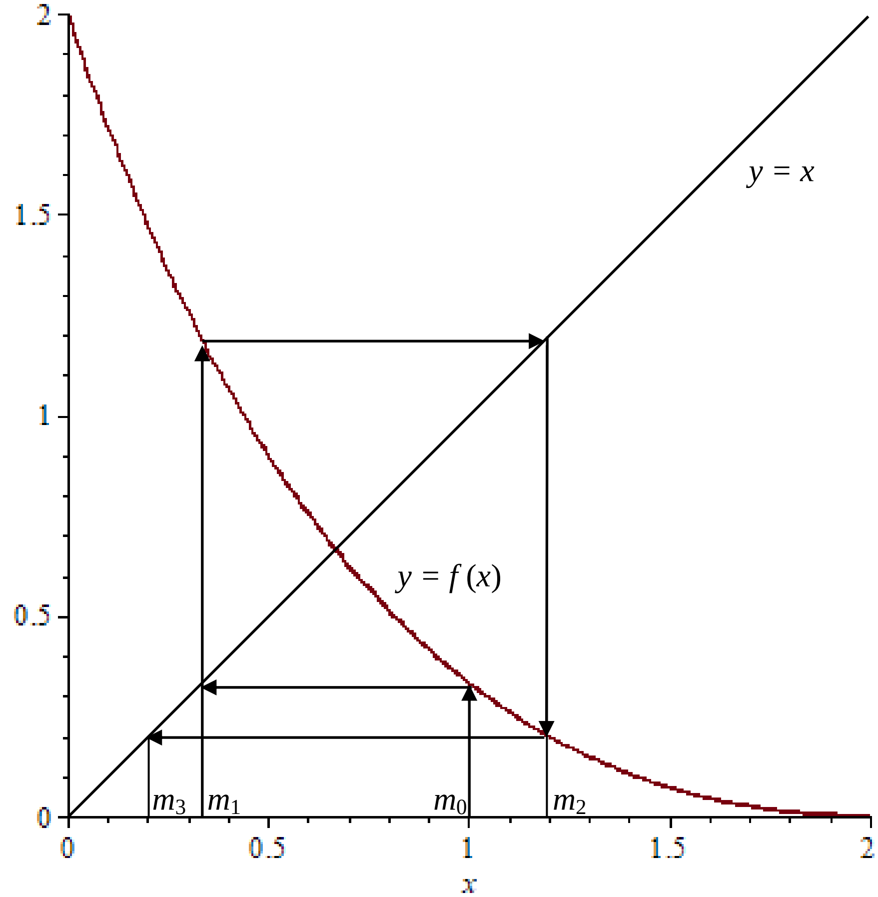
Part II. Detailed-answer Questions (10 points)

6.1. (5 points) Suppose a discrete-time dynamic system (DTDS) has recursive relation
$m_{i+1} = \left(\frac{m_i - 6}{m_i + 2}\right) m_i + 2$, $m_i \geq 0$.

(a) (2 points) Find the equilibrium of this system.

Solution. The updating function of this system is $f(x) = \left(\frac{x-6}{x+2}\right)x + 2$. Let $x = \left(\frac{x-6}{x+2}\right)x + 2$.
Then $x(x + 2) = (x - 6)x + 2(x + 2)$, $x^2 + 2x = x^2 - 4x + 4$, $6x = 4$, $x = 2 / 3$. The only equilibrium of this system is $x = 2 / 3$.

(b) (2 points) The following figure shows the graph of the updating function of this system. Work on this figure using cobwebbing starting from $m_0 = 1$ to plot m_1, m_2, and m_3 on the horizontal axis.



(c) (1 point) Is this equilibrium stable or unstable?

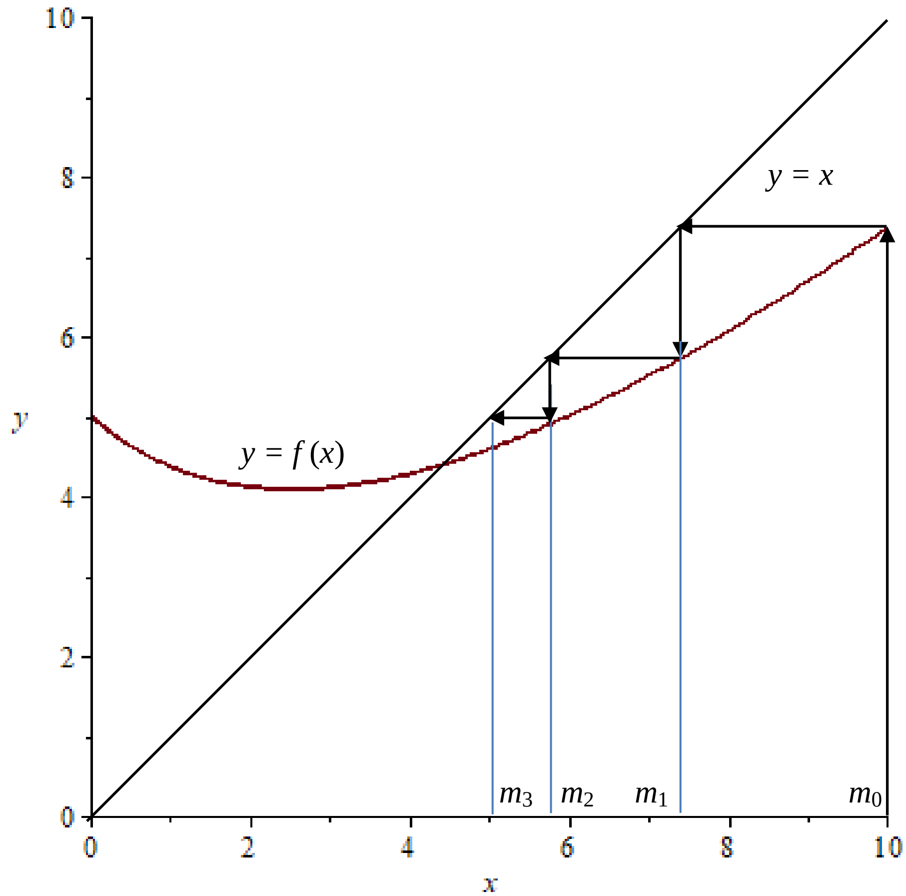
Answer. This equilibrium is unstable.

6.2. (5 points) Suppose a discrete-time dynamic system (DTDS) has recursive relation $m_{i+1} = \left(\dfrac{m_i - 6}{m_i + 7}\right) m_i + 5$, $m_i \geq 0$.

(a) (2 points) Find the equilibrium of this system.

Solution. The updating function of this system is $f(x) = \left(\dfrac{x-6}{x+7}\right)x + 5$. Let $x = \left(\dfrac{x-6}{x+7}\right)x + 5$. Then $x(x + 7) = (x - 6)x + 5(x + 7)$, $x^2 + 7x = x^2 - x + 35$, $8x = 35$, $x = 35 / 8$. The only equilibrium of this system is $x = 35 / 8$.

(b) (2 points) The following figure shows the graph of the updating function of this system. Work on this figure using cobwebbing starting from $m_0 = 10$ to plot m_1, m_2, and m_3 on the horizontal axis.



(c) (1 point) Is this equilibrium stable or unstable?

Answer. This equilibrium is stable.

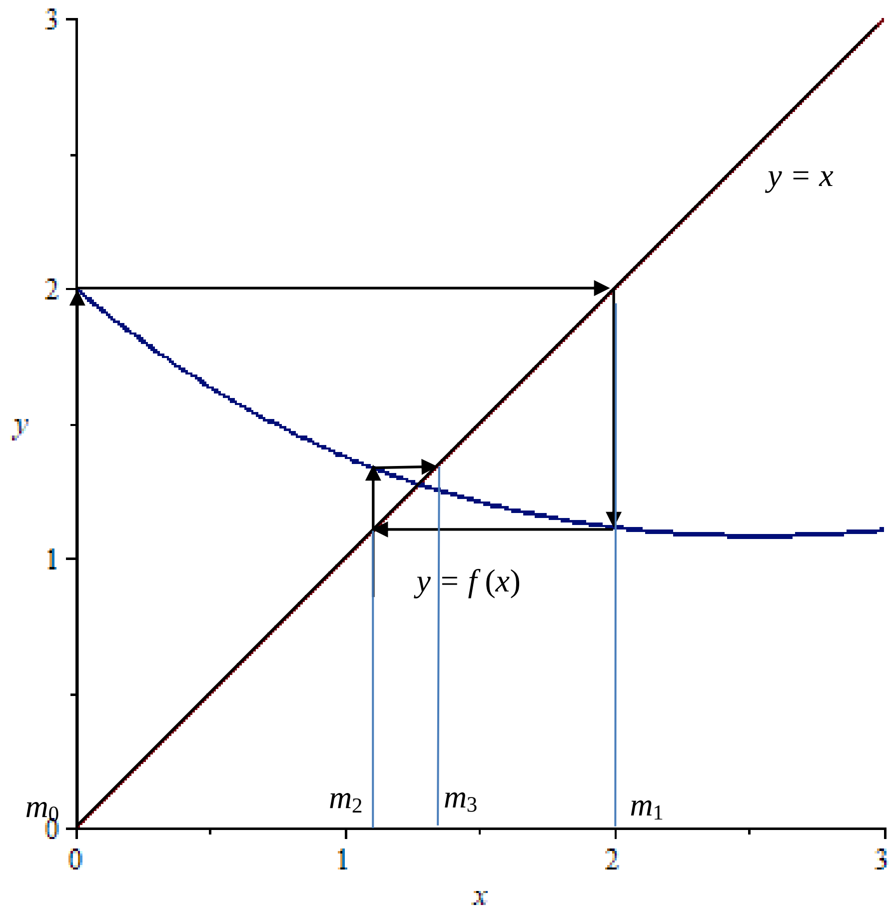
6.3. (5 points) Suppose a discrete-time dynamic system (DTDS) has recursive relation

$m_{i+1} = \left(\frac{m_i - 6}{m_i + 7}\right) m_i + 2$, $m_i \geq 0$.

(a) (2 points) Find the equilibrium of this system.

Solution. The updating function of this system is $f(x) = \left(\frac{x-6}{x+7}\right)x+2$. Let $x = \left(\frac{x-6}{x+7}\right)x+2$.
Then $x(x + 7) = (x - 6)x + 2(x + 7)$, $x^2 + 7x = x^2 - 4x + 14$, $11x = 14$, $x = 14 / 11$. The only equilibrium of this system is $x = 14 / 11$.

(b) (2 points) The following figure shows the graph of the updating function of this system. Work on this figure using cobwebbing starting from $m_0 = 0$ to plot m_1, m_2, and m_3 on the horizontal axis.



(c) (1 point) Is this equilibrium stable or unstable?

Answer. This equilibrium is stable.

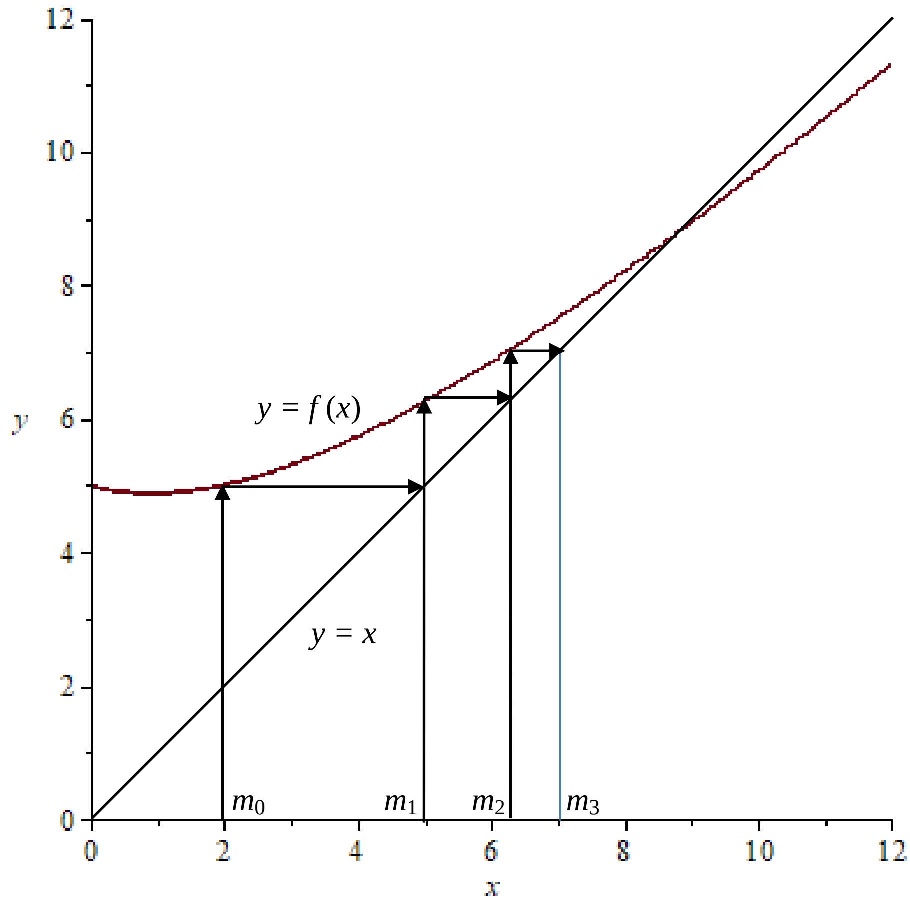
6.4. (5 points) Suppose a discrete-time dynamic system (DTDS) has recursive relation

$m_{i+1} = \left(\frac{m_i - 2}{m_i + 7}\right) m_i + 5$, $m_i \geq 0$.

(a) (2 points) Find the equilibrium of this system.

Solution. The updating function of this system is $f(x) = \left(\frac{x-2}{x+7}\right)x + 5$. Let $x = \left(\frac{x-2}{x+7}\right)x + 5$. Then $x(x + 7) = (x - 2)x + 5(x + 7)$, $x^2 + 7x = x^2 + 3x + 14$, $4x = 35$, $x = 35 / 4$. The only equilibrium of this system is $x = 35 / 4$.

(b) (2 points) The following figure shows the graph of the updating function of this system. Work on this figure using cobwebbing starting from $m_0 = 2$ to plot m_1, m_2, and m_3 on the horizontal axis.



(c) (1 point) Is this equilibrium stable or unstable?

Answer. This equilibrium is stable.

7.1. (5 points) (a) (2 points) What is the definition of the derivative of a function $y = f(x)$ at a point $x = a$?

Answer. $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, or $f'(a) = \lim_{b\to a}\frac{f(b)-f(a)}{h}$.

(b) (3 points) Find the derivative of the function $f(x) = \frac{x}{x+1}$ at the point $x = 1$ **by definition**.

Solution.

$$f'(1) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1+h}{(1+h)+1}-\frac{1}{2}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{2(1+h)-(1+h+1)}{2((1+h)+1)}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{h}{4+2h}\right) = \lim_{h\to 0}\frac{1}{4+2h} = \frac{1}{4}.$$

7.2. (5 points) (a) (2 points) What is the definition of the derivative of a function $y = f(x)$ at a point $x = a$?

Answer. $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, or $f'(a) = \lim_{b\to a}\frac{f(b)-f(a)}{h}$.

(b) (3 points) Find the derivative of the function $f(x) = \frac{x}{x-1}$ at the point $x = 2$ **by definition**.

Solution.

$$f'(2) = \lim_{h\to 0}\frac{1}{h}\left(\frac{2+h}{(2+h)-1}-\frac{2}{1}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{(2+h)-2(1+h)}{1+h}\right) = -\lim_{h\to 0}\frac{1}{h}\left(\frac{h}{1+h}\right) = -\lim_{h\to 0}\frac{1}{1+h} = -1.$$

7.3. (5 points) (a) (2 points) What is the definition of the derivative of a function $y = f(x)$ at a point $x = a$?

Answer. $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, or $f'(a) = \lim_{b\to a}\frac{f(b)-f(a)}{h}$.

(b) (3 points) Find the derivative of the function $f(x) = \frac{x}{x+2}$ at the point $x = 1$ **by definition**.

Solution.

$$f'(1)=\lim_{h\to 0}\frac{1}{h}\left(\frac{1+h}{(1+h)+2}-\frac{1}{3}\right)=\lim_{h\to 0}\frac{1}{h}\left(\frac{3(1+h)-(3+h)}{3(3+h)}\right)=\lim_{h\to 0}\frac{1}{h}\left(\frac{2h}{9+3h}\right)=\lim_{h\to 0}\frac{2}{9+3h}=\frac{2}{9}\,.$$

7.4. (5 points) (a) (2 points) What is the definition of the derivative of a function $y = f(x)$ at a point $x = a$?

Answer. $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, or $f'(a) = \lim_{b\to a}\frac{f(b)-f(a)}{h}$.

(b) (3 points) Find the derivative of the function $f(x) = \frac{x}{x-2}$ at the point $x = 3$ **by definition**.

Solution.

$$f'(3)=\lim_{h\to 0}\frac{1}{h}\left(\frac{3+h}{(3+h)-2}-\frac{3}{1}\right)=\lim_{h\to 0}\frac{1}{h}\left(\frac{(3+h)-3(1+h)}{1+h}\right)=\lim_{h\to 0}\frac{1}{h}\left(\frac{-2h}{1+h}\right)=-2\lim_{h\to 0}\frac{1}{1+h}=-2\,.$$