

# Review of Linear Systems

- Linear system -> the superposition principle holds:

$$\mathbf{L}[a_1x_1(t) + a_2x_2(t)] = a_1\mathbf{L}[x_1(t)] + a_2\mathbf{L}[x_2(t)]$$

- Time-invariant system -> shift in time does not change the response:

$$y(t) = \mathbf{L}[x_1(t)] \rightarrow y(t - t_0) = \mathbf{L}[x_1(t - t_0)], \forall t_0$$

i.e., the operator  $\mathbf{L}[\ ]$  does not change in time

- Linear time-invariant (LTI) system -> both, linear and time-invariant

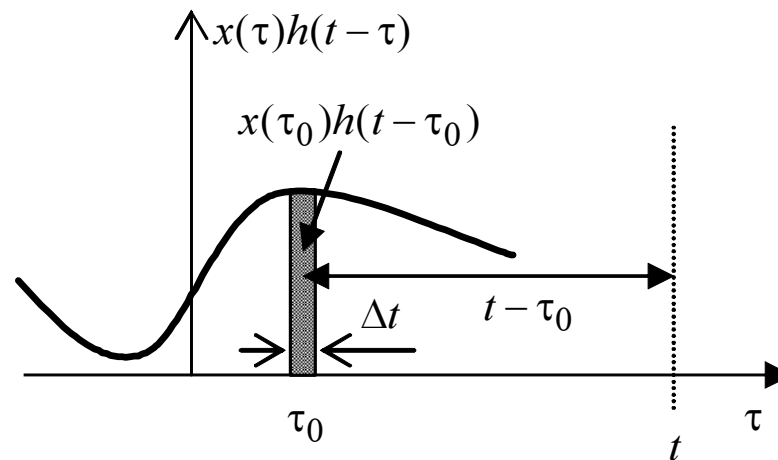
# Response of LTI System

- If the input is a Dirac delta function, the output is impulse response:

$$h(t) = \mathbf{L}[\delta(t)]$$

- If the input is  $x(t)$ , the output is a convolution integral:

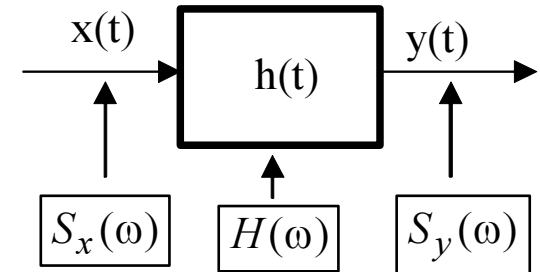
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = x(t) * h(t)$$



# Transfer Function

- Convolution theorem in frequency domain:

$$y(t) = x(t) * h(t) \leftrightarrow S_y(f) = S_x(f)H(f)$$



- Transfer function (frequency response):

$$H(f) = \frac{S_y(f)}{S_x(f)}$$

$$h(t) \leftrightarrow H(f) = \mathbf{L}[1]$$

- Amplitude response & phase response:

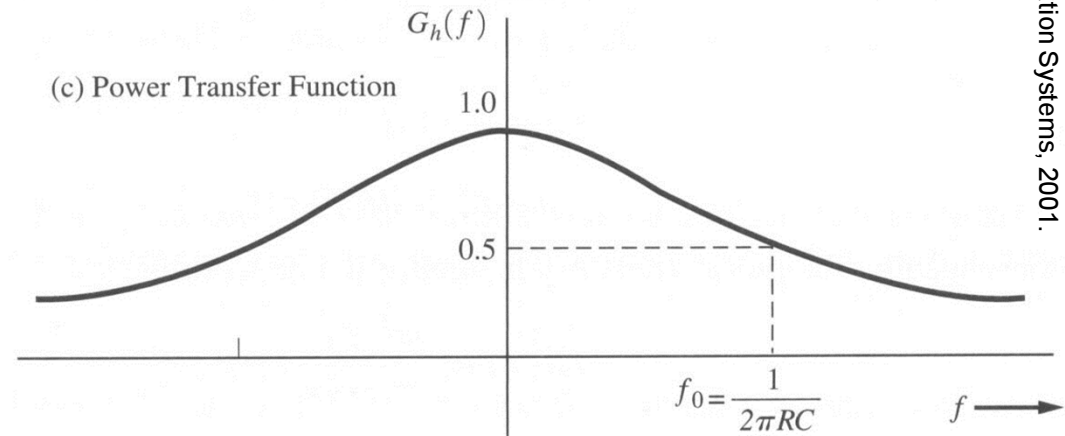
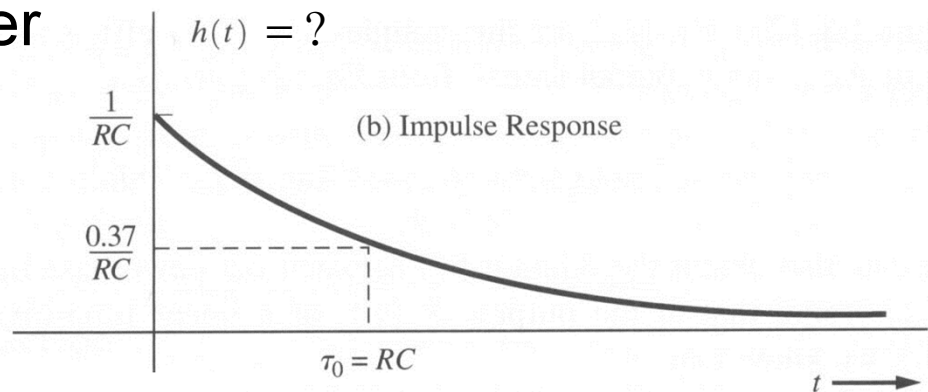
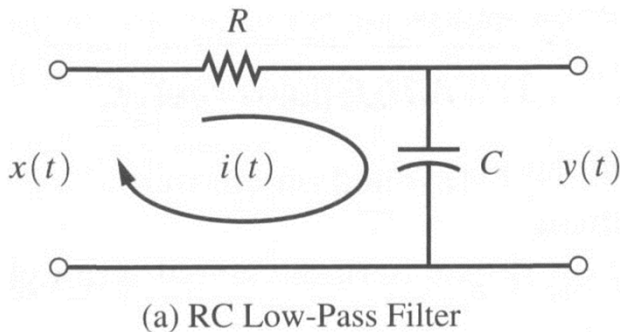
$$H(f) = |H(f)|e^{j\varphi(f)} = |H(f)|\angle\varphi(f)$$

$$\varphi(f) = \tan^{-1} \left( \frac{\text{Im} \{H(f)\}}{\text{Re} \{H(f)\}} \right)$$

# Power Transfer Function

- Power transfer function:  $G(f) = \frac{P_y(f)}{P_x(f)} = |H(f)|^2$

- Example 2.14: RC filter



L.W. Couch, II, Digital and Analog Communication Systems, 2001.

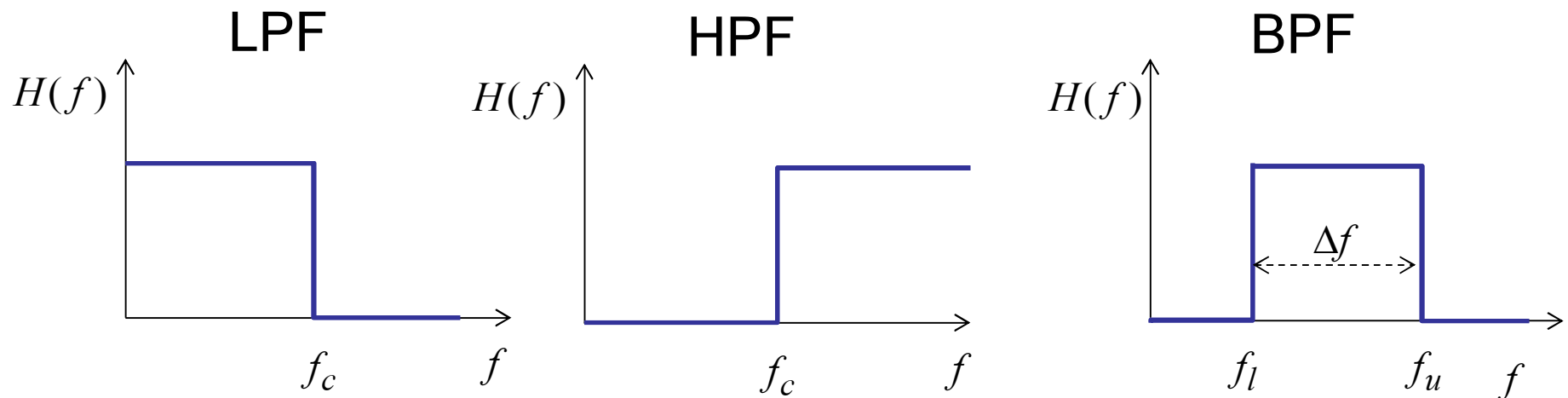
Derive it.

$$H(f) = \frac{1}{1 + jf / f_0}$$

$$G(f) = \frac{1}{1 + (f / f_0)^2}$$

# Ideal (“brick-wall”) Filters

- low-pass filter (LPF)
- high-pass filter (HPF)
- band-pass filter (BPF)



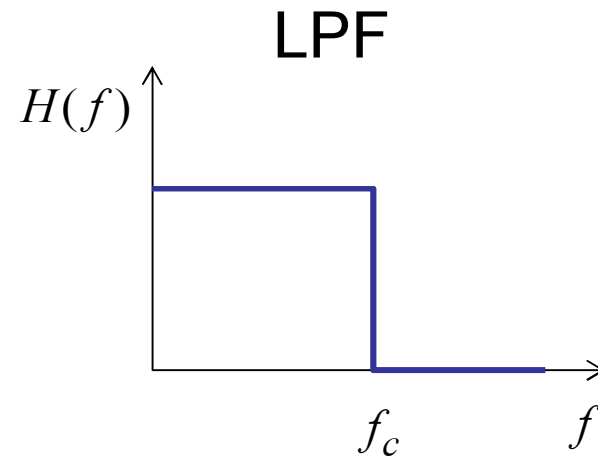
# Ideal Filters: LPF

- Ideal (“brick-wall”) low-pass filter (LPF)
  - Frequency response:

$$H(f) = \begin{cases} 1, & |f| \leq f_c \\ 0, & |f| > f_c \end{cases}$$

- impulse response:

$$h(t) = ?$$

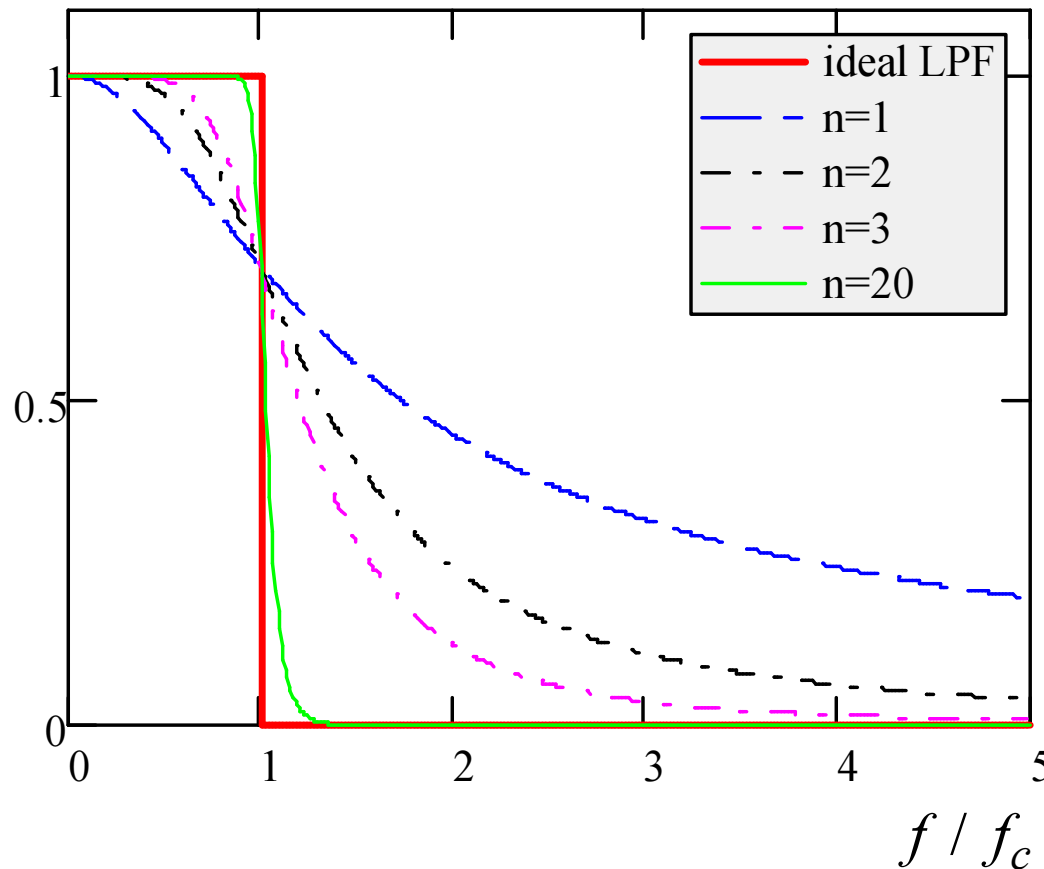


# Ideal Filters: HPF, BPF

- HPF:  $H(f)=?$   $h(t)=?$
- BPF:  $H(f)=?$   $h(t)=?$
  
- How to build BPF using LPF and HPF?
  - suggest at least 2 ways
  
- Practical: measured in Lab 1

# Practical Filters: Butterworth

- Frequency response:  $|H_n(f)| = \frac{1}{\sqrt{1 + (f / f_c)^{2n}}}$



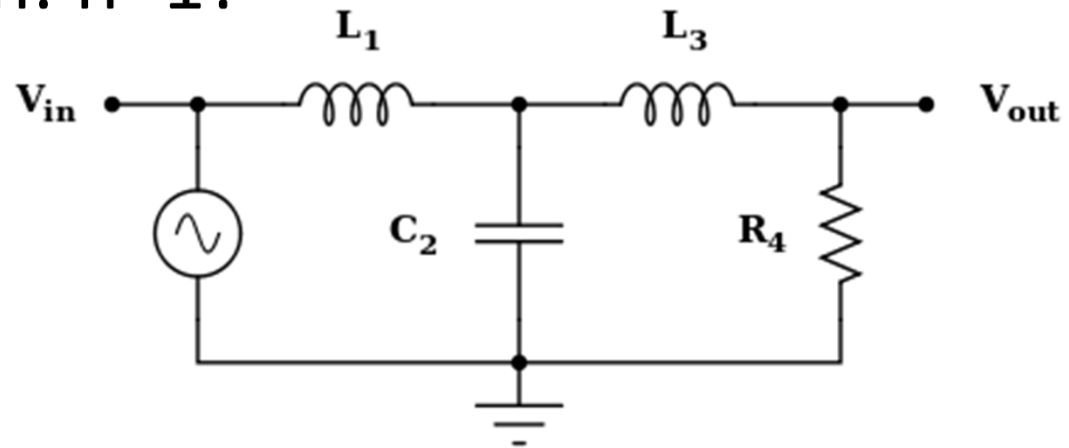
Q: show that

$$\lim_{n \rightarrow \infty} |H_n(f)| = H_{\text{LPF}}(f)$$

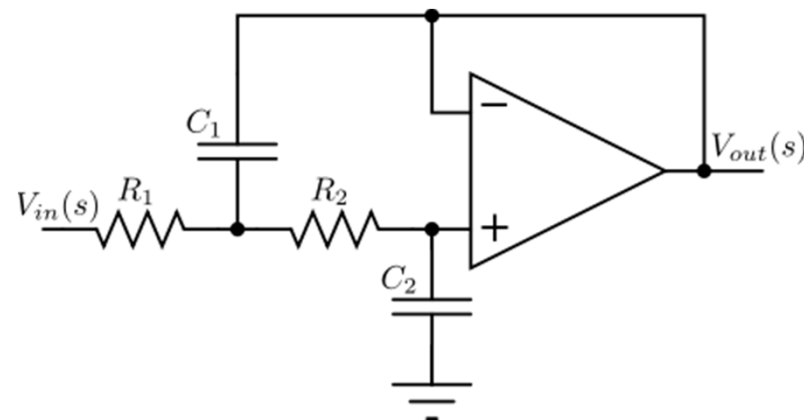
# Practical Filters: Butterworth

- Implementation:  $n=1$ ?

- $n=3$ , passive

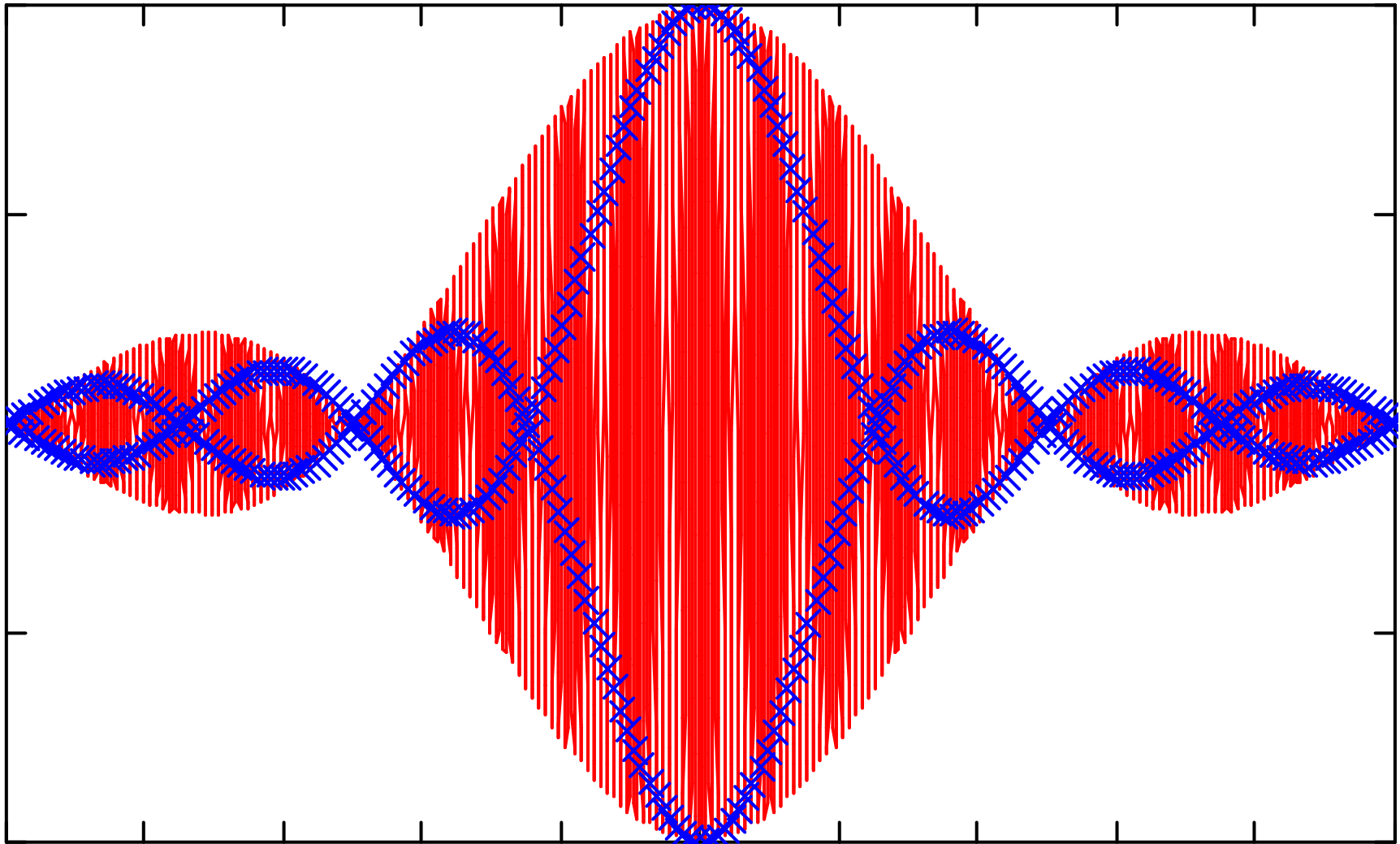


- $n=2$ , active



Adopted from "Wikipedia: Butterworth filter"

???

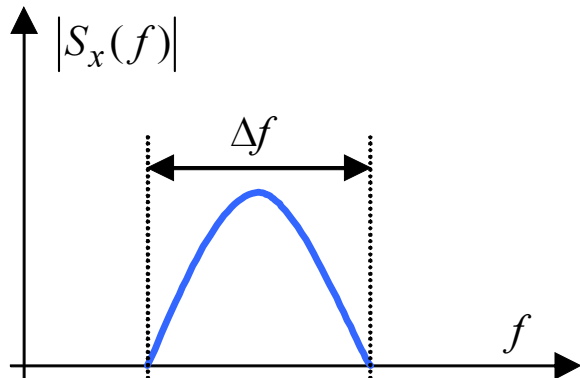


# Signal Bandwidth

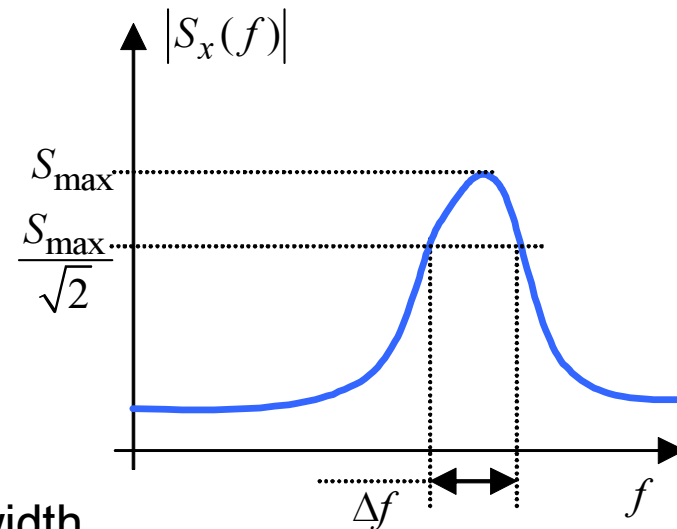
- Defined for positive frequencies only.
- Informal: a frequency band over which a substantial (or all) signal power is concentrated
- Absolute bandwidth: for band-limited signals, frequency band where spectrum is not zero. For all other frequencies, the spectrum must be zero.
- 3 dB (half-power) bandwidth: frequency band where PSD (or ESD) is not lower than -3 dB with respect to maximum
- Zero-crossing bandwidth: frequency band limited by 1st zero(s) in the spectrum.

# Signal Bandwidth

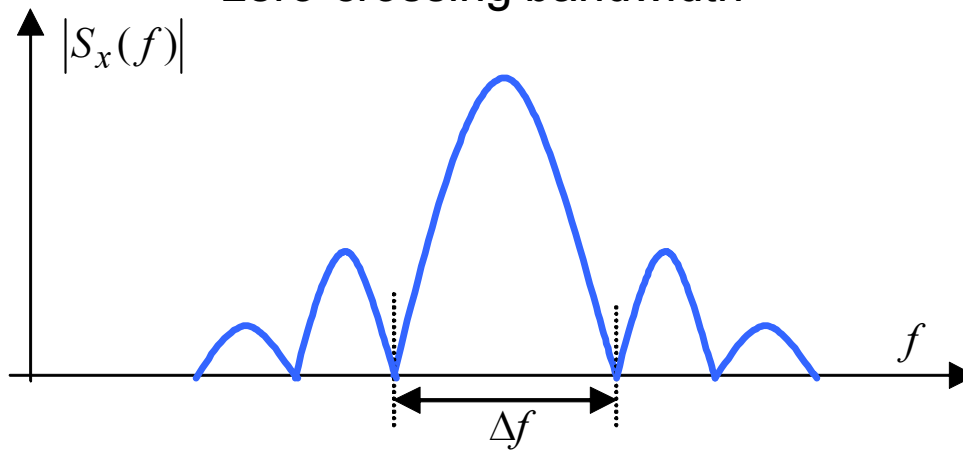
absolute bandwidth



3 dB bandwidth



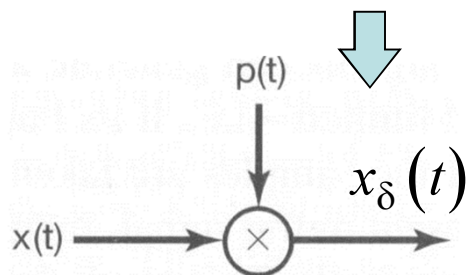
zero-crossing bandwidth



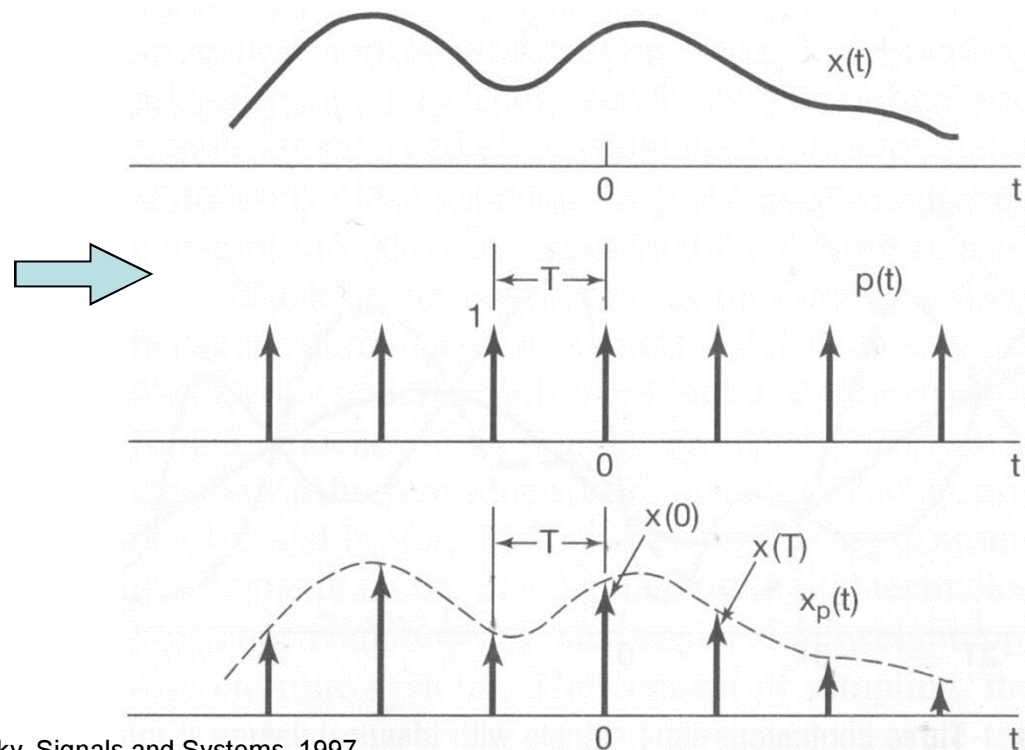
# Sampling Theorem

- The sampling theorem is one of the most important results. A bridge between digital and analog worlds. DSP is based on it.
- Intuitive explanation: interpolation.
- Signal sampling:

$$x_{\delta}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s)$$



A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.



# Sampling Theorem

- Let  $x(t)$  be bandlimited,  $S_x(f) = 0, \forall |f| \geq f_{\max}$
- Sample it at multiples of sampling interval  $T_s$ ,

$$\{x(nT_s)\}_{n=-\infty}^{+\infty}, T_s \leq 1/(2f_{\max})$$

- $x(t)$  is uniquely determined by its samples  $\{x(nT_s)\}_{n=-\infty}^{+\infty}$
- $x(t)$  can be reconstructed as follows:

$$x(t) = 2fT_s \sum_{n=-\infty}^{\infty} x(nT_s) \text{sinc}[2f'(t - nT_s)]$$

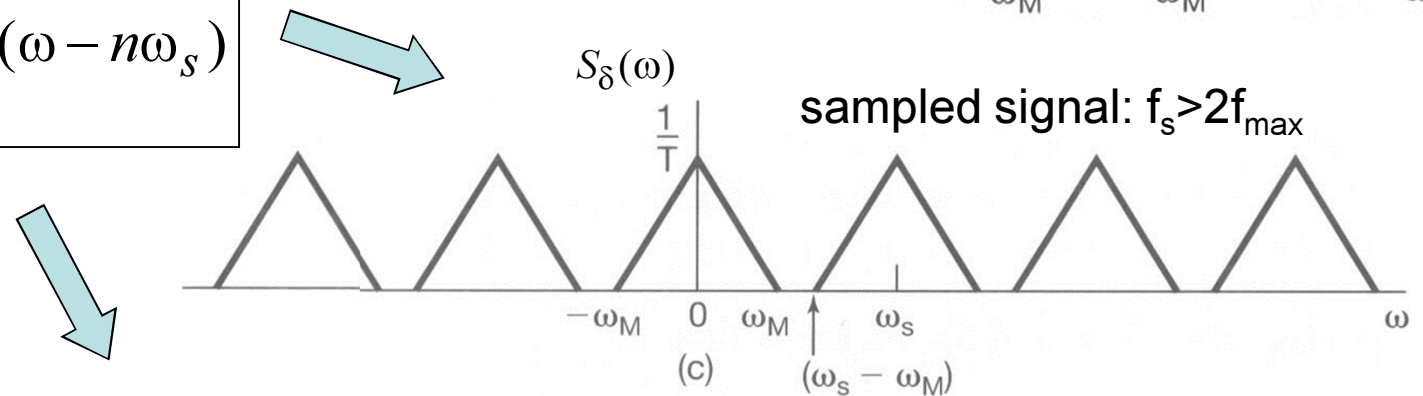
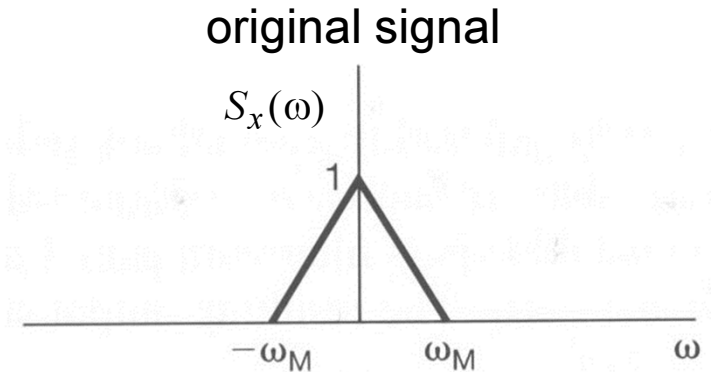
$$\xrightarrow{f'=?} \sum_{n=-\infty}^{\infty} x(nT_s) \text{sinc}[t/T_s - n]$$

where  $f_{\max} \leq f' \leq \frac{1}{T_s} - f_{\max}$ ,  $f_s = \frac{1}{T_s} \geq 2f_{\max}$  - sampling frequency  
 $2f_{\max}$  - Nyquist frequency

# Sampling Theorem

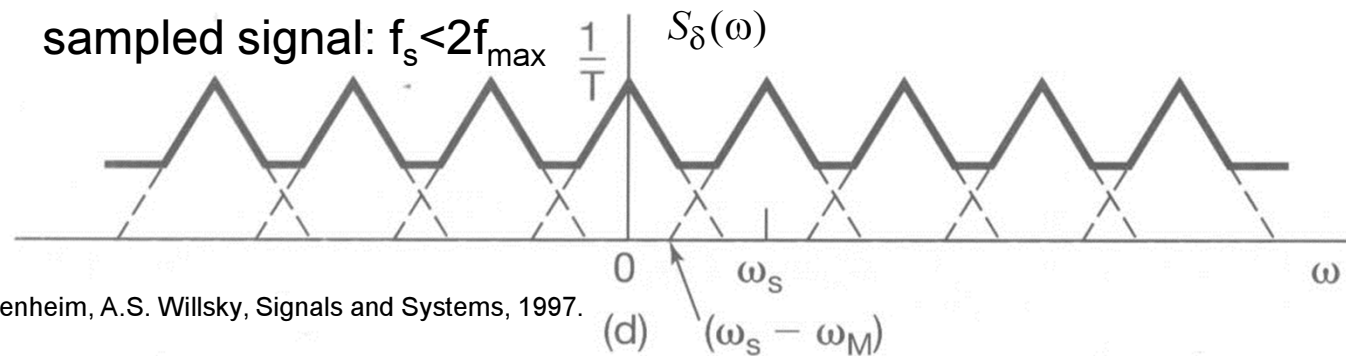
$$x_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$\begin{aligned} x_\delta(t) &\leftrightarrow S_x(\omega) * S_\delta(\omega) \\ &= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} S_x(\omega - n\omega_s) \end{aligned}$$



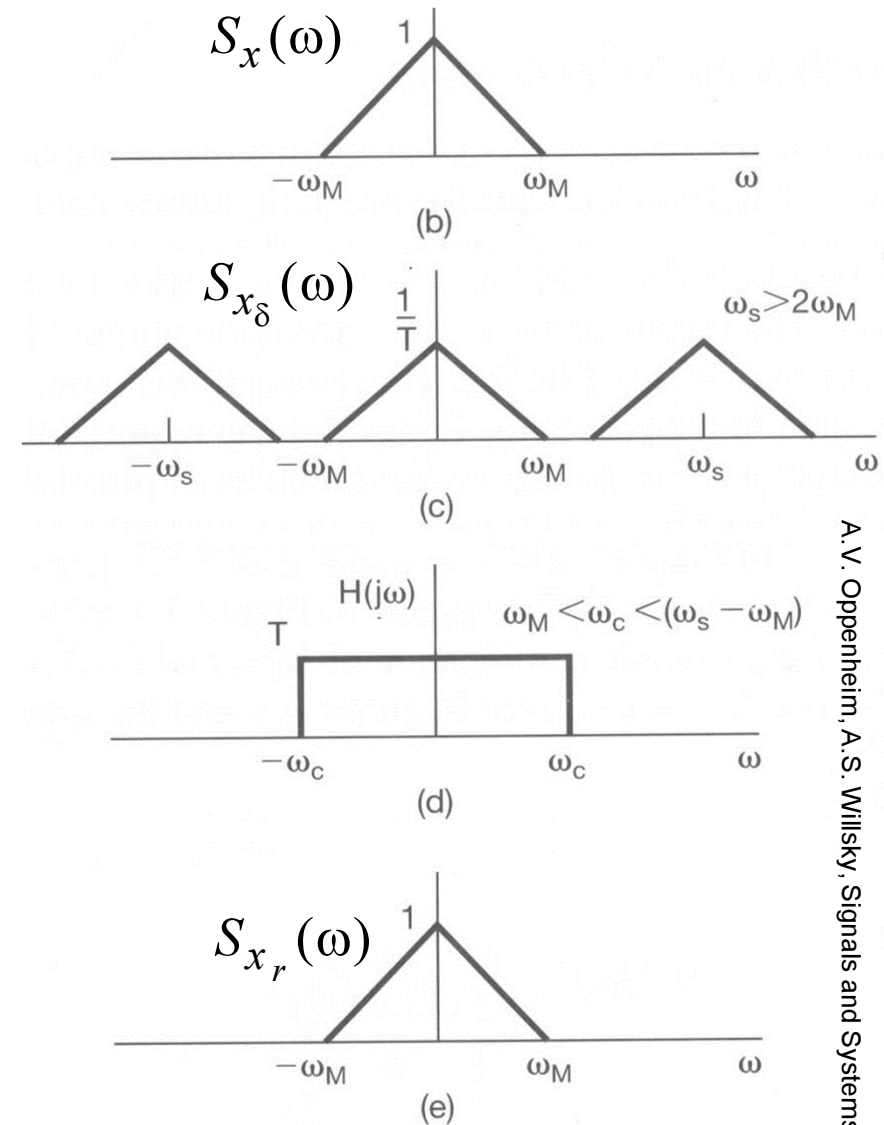
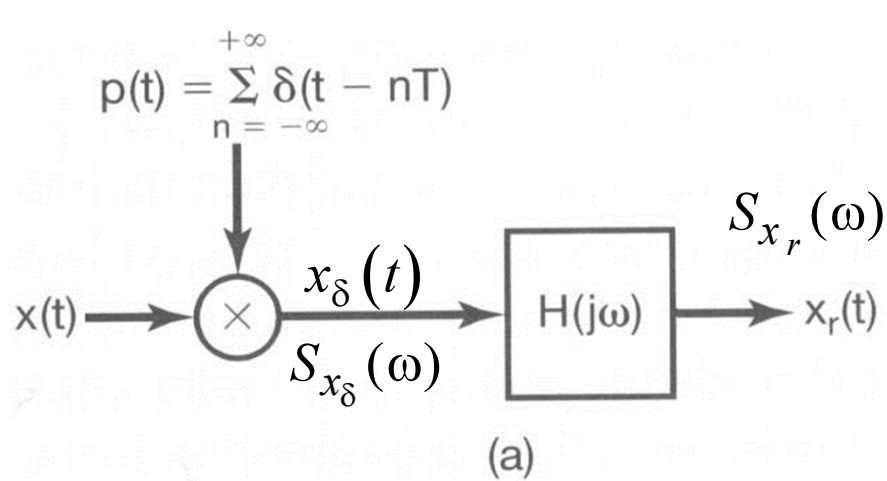
Nyquist frequency:

$$f_{s,\min} = 2f_{\max}$$



A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.

# Recovery of Sampled Signal



- sample the signal with  $f_s > 2f_{\max}$
- use low-pass filter with  $f_{\max} < f_c < f_s - f_{\max}$  to recover original signal

# Dimensionality Theorem

- A real signal of bandwidth  $B$  can be recovered from

$$N = 2BT$$

samples over a time interval  $T$  provided that  $BT \gg 1$

- This is a foundation for digital communications, signal processing, Internet, etc.
- If not band-limited?
- Sampling in practice:  $f_s = 2B + \Delta f$
- **Example:** 1h of HiFi music,  $N=?$

# On Time/Bandlimited Signals

- An absolutely bandlimited signal cannot be time limited and vice versa.
- Engineering implications?

# Summary

- Review of linear systems. Response of LTI system in time & frequency domains. Power transfer function.
- Signal bandwidth.
- Sampling theorem. Recovery of sampled signals.
- Good reference on filters (etc.): W. Siebert, Circuits, signals, and systems, McGraw Hill, ch. 15.
- **Homework**: Couch, 2.6, 2.7, 2.9-2.11; Oppenheim & Willsky, Ch. 2.2, 2.3, 6.1-6.4, 7.1-7.3. Study carefully all the examples (including end-of-chapter study-aid examples), make sure you understand and can solve them with the book closed.
- Do some end-of-chapter problems. Students' solution manual provides solutions for many of them.