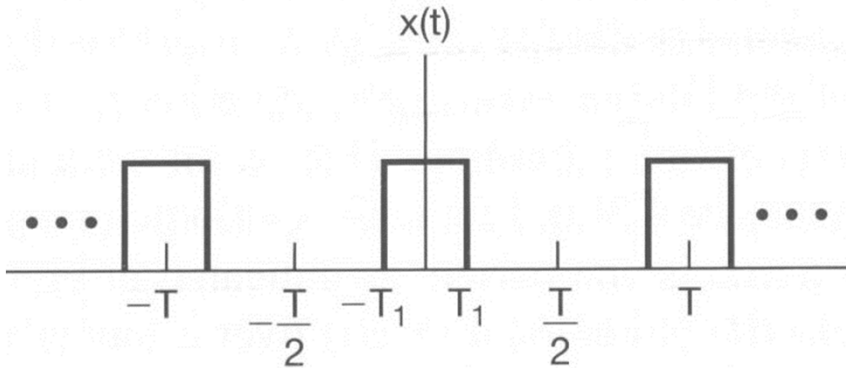


# Review of Fourier Transform

- Fourier series works for periodic signals only. What's about aperiodic signals? This is very large & important class of signals
- Aperiodic signal can be considered as periodic with  $T \rightarrow \infty$
- Fourier series changes to Fourier transform, complex exponents are infinitesimally close in frequency
- Discrete spectrum becomes a continuous one, also known as spectral density

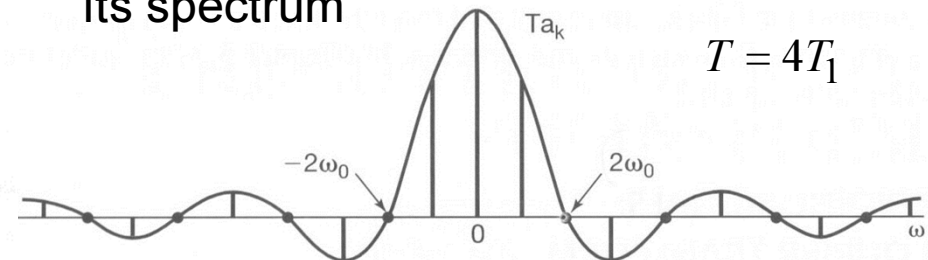
# Fourier Series -> Fourier Transform

Periodic signal

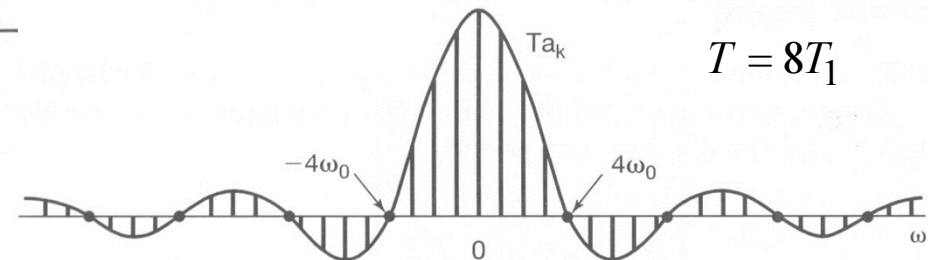


As  $T$  increases, spectral components are getting closer and closer, becoming the continuous spectrum at the limit

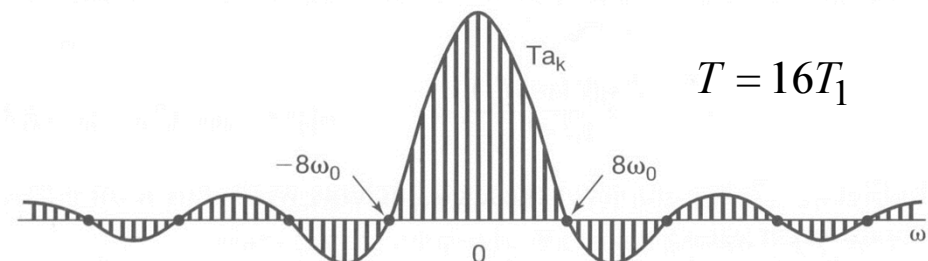
Its spectrum



(a)



(b)



(c)

A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.

# Fourier Transform

- Fourier transform (spectrum): radial frequency

$$S_x(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt$$

$$S_x(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

- Inverse Fourier transform: radial frequency

$$x(t) = \int_{-\infty}^{+\infty} S_x(f) e^{j2\pi ft} df$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_x(\omega) e^{j\omega t} d\omega$$

- Dirichlet conditions (details on the next page):
  - $x(t)$  is absolutely integrable,
  - $x(t)$  has a finite number of maxima & minima within any finite interval,
  - $x(t)$  has a finite number of discontinuities within any finite interval. These discontinuities must be finite.

# Convergence of Fourier Transform

- Dirichlet conditions:

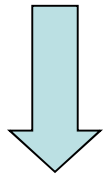
- $x(t)$  must be absolutely integrable (finite energy)

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty \quad \text{or} \quad \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

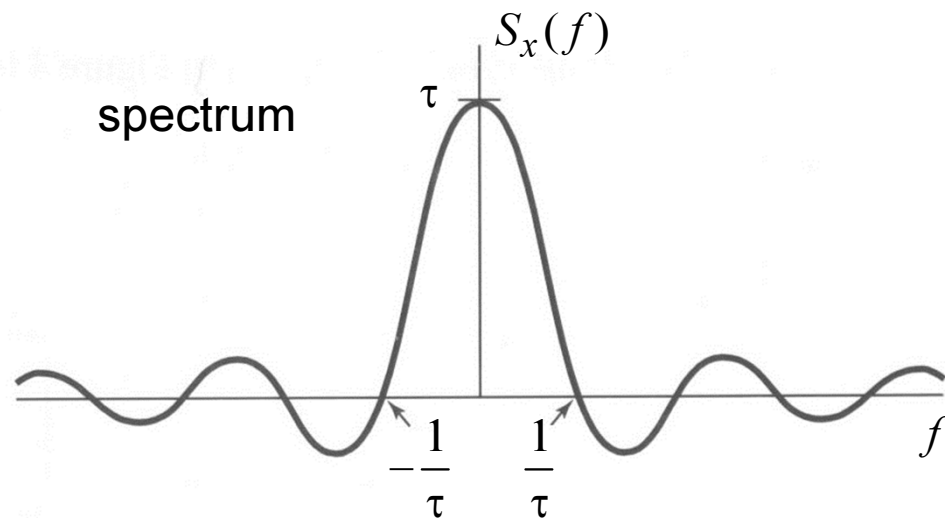
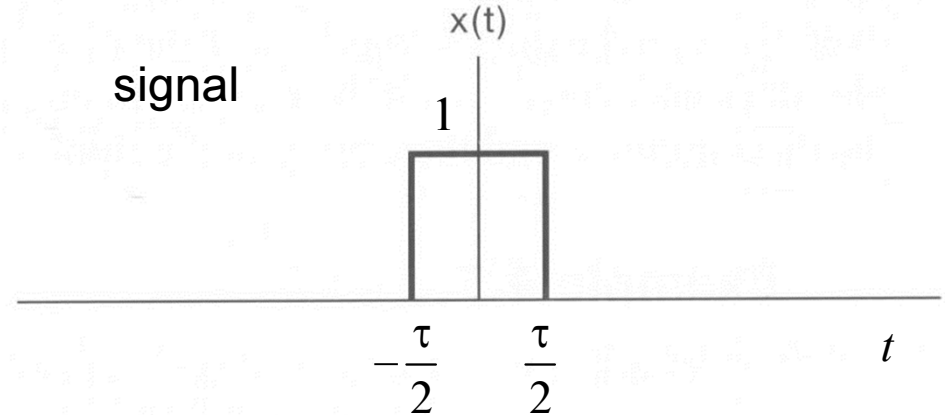
- $x(t)$  has a finite number of maxima & minima within any finite interval,
  - $x(t)$  has a finite number of discontinuities within any finite interval. These discontinuities must be finite.
- Dirichlet conditions are only sufficient, but are not necessary.
  - All physically-reasonable (practical) signals meet these conditions.

# Example: Rectangular Pulse

$$x(t) = \Pi(t/\tau) = \begin{cases} 1, & |t| < \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

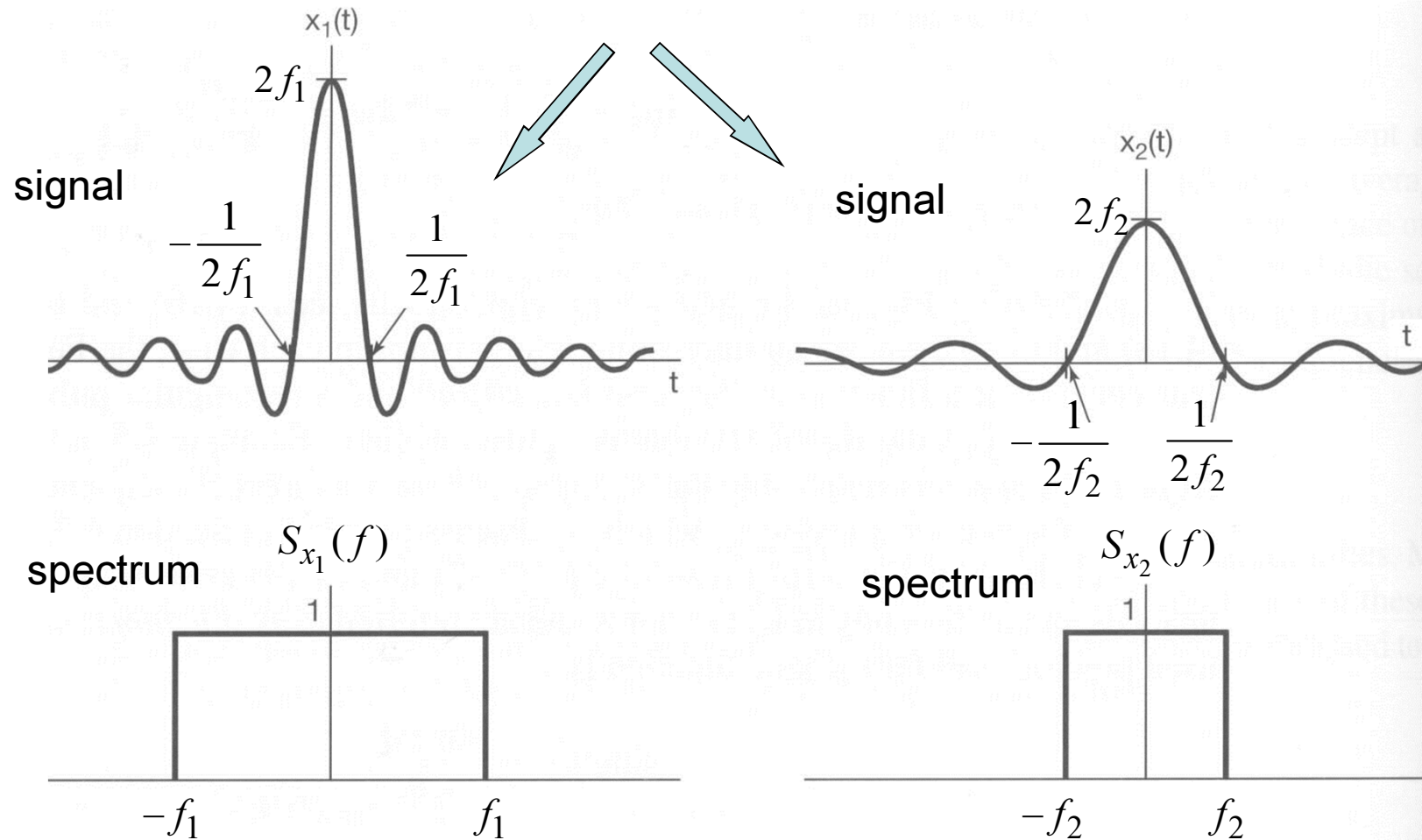


$$S_x(f) = \tau \frac{\sin \pi f \tau}{\pi f \tau} = \tau \cdot \text{sinc}(f \tau)$$



# Example: sinc(t)

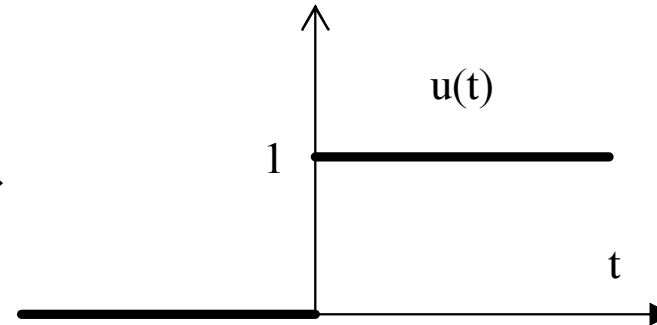
Shortening pulse widens its spectrum!



# Generalized (Singular) Functions

- Unit step function

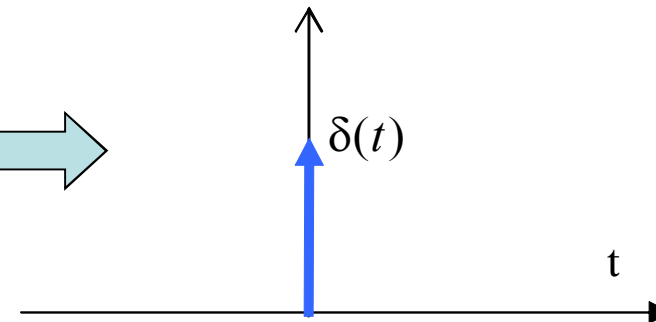
$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$



- Dirac delta function (unit impulse function)

$$\int_{-\infty}^{\infty} \delta(t)x(t)dt = x(0)$$

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \quad \& \quad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

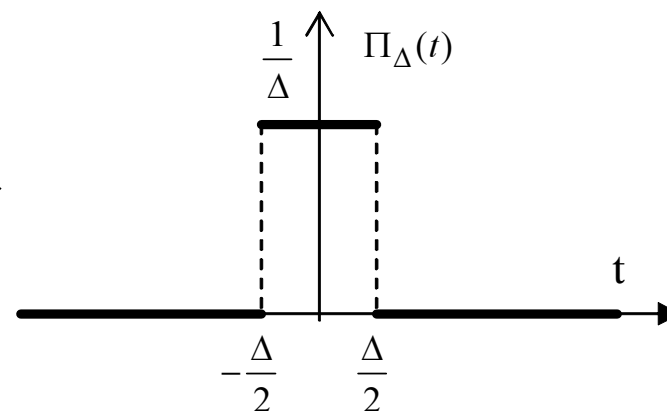


# Generalized Functions

- Dirac Delta Function as a limit:

$$\Pi_{\Delta}(t) = \begin{cases} 1/\Delta, & |t| < \Delta/2 \\ 0, & |t| > \Delta \end{cases}$$

$$\delta(t) = \lim_{\Delta \rightarrow 0} \Pi_{\Delta}(t)$$



- Useful properties of delta function

Fourier transform?

$$\int_{-\infty}^{\infty} \delta(t - t_0) x(t) dt = x(t_0)$$

$$\int_{-\infty}^t \delta(t) dt = u(t)$$

$$\frac{du(t)}{dt} = \delta(t)$$

$$\delta(at) = \frac{1}{|a|} \delta(t)$$

$$\delta(-t) = \delta(t)$$

$$x(t)\delta(t) = x(0)\delta(t)$$

# Fourier Transform of Periodic Signal

- FT of a complex exponent:

$$x(t) = e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0) = \delta(f - f_0)$$

- Important property:

$$\delta(f) = \int_{-\infty}^{+\infty} e^{\pm j2\pi ft} dt \quad \leftarrow \text{prove this property}$$

- FT of a periodic signal:

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} \xrightarrow{FT} 2\pi \sum_{n=-\infty}^{+\infty} c_n \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{+\infty} c_n \delta(f - nf_0)$$

- FT of  $\cos(\omega_0 t)$  ?

# Properties of Fourier Transform\*

- Very similar to those of Fourier series!

- Linearity:

$$\alpha x_1(t) + \beta x_2(t) \xleftrightarrow{F} \alpha S_{x_1}(f) + \beta S_{x_2}(f)$$

- Time shifting:

$$x(t) \leftrightarrow S_x(\omega) \Rightarrow x(t - t_0) \leftrightarrow e^{-j\omega t_0} S_x(\omega)$$

- Time reversal:

$$x(t) \leftrightarrow S_x(\omega) \Rightarrow x(-t) \leftrightarrow S_x(-\omega)$$

- Time scaling:

$$x(at) \leftrightarrow \frac{1}{|a|} S_x\left(\frac{\omega}{a}\right)$$

Prove these properties.

\*properties are useful for evaluating Fourier transform in a simple way

# Properties of Fourier Transform

- Conjugation:  $x(t) \leftrightarrow S_x(\omega) \Rightarrow x^*(t) \leftrightarrow S_x^*(-\omega)$

- Differentiation:  $x(t) \leftrightarrow S_x(\omega) \Rightarrow \frac{dx(t)}{dt} \leftrightarrow j\omega S_x(\omega)$

- Integration:  $\int_{-\infty}^t x(t)dt \leftrightarrow \frac{1}{j\omega} S_x(\omega) + \pi S_x(0)\delta(\omega)$

- Multiplication:  $x(t)y(t) \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega')S_y(\omega - \omega')d\omega' = S_x(\omega) * S_y(\omega)$

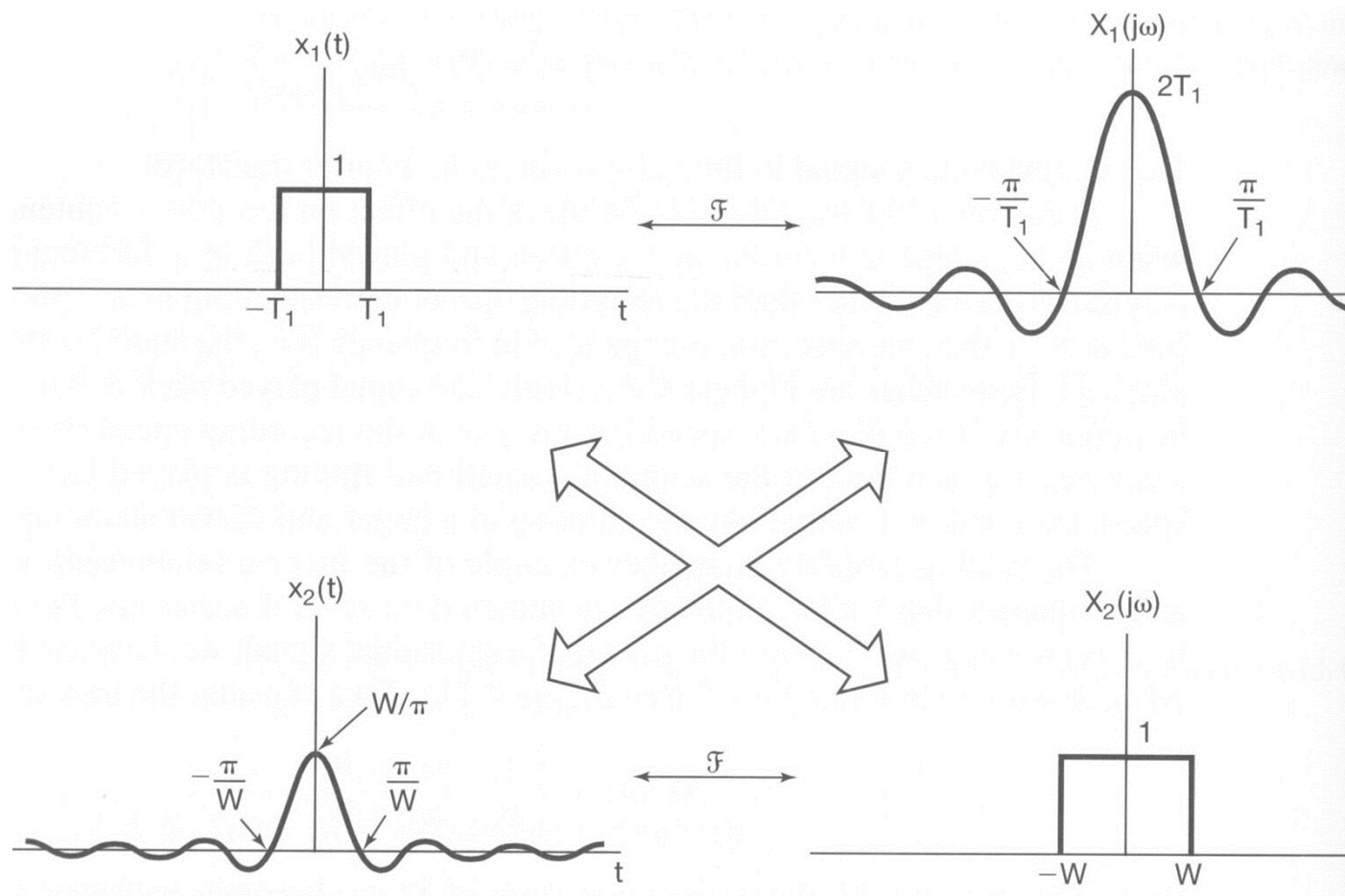
- Frequency shift (modulation):  $x(t)e^{j\omega_0 t} \leftrightarrow S(\omega - \omega_0)$

Prove these properties

# Duality of Fourier Transform

$$x(t) \leftrightarrow S_x(\omega) \Rightarrow S_x(t) \leftrightarrow 2\pi x(-\omega)$$

$$x(t) \leftrightarrow S_x(f) \Rightarrow S_x(t) \leftrightarrow x(-f)$$

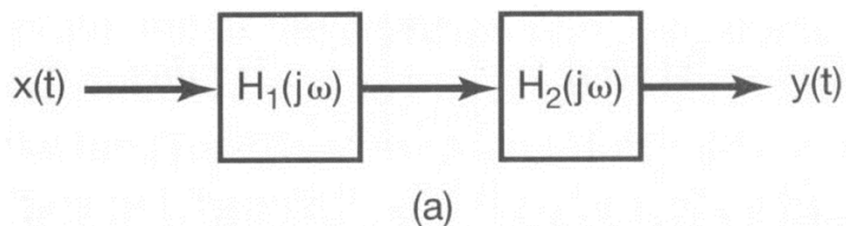
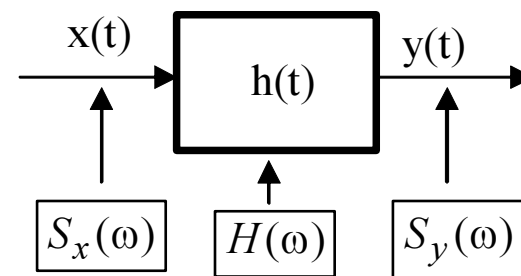


A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.

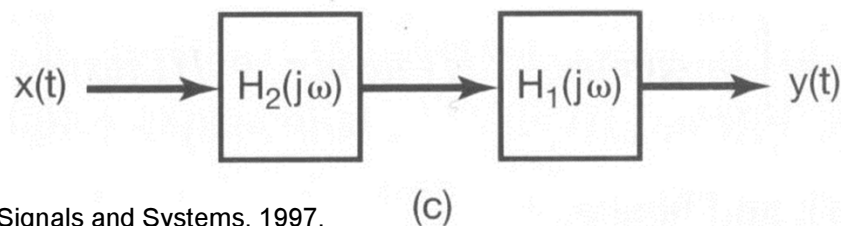
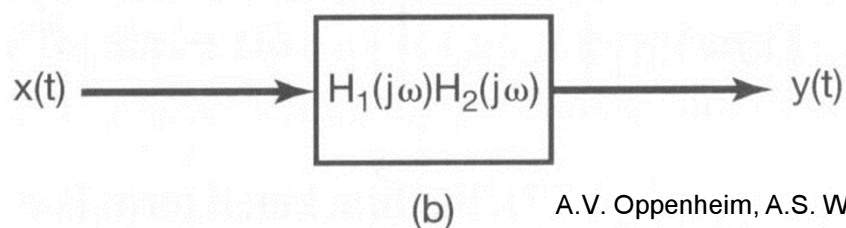
# Convolution Property

- This property is of great importance

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \leftrightarrow S_x(\omega)H(\omega) = S_y(\omega)$$



← Cascade connection of LTI blocks



A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.

- Example: FT of a triangular pulse by convolution

# Parseval Theorem

- Total energy in time domain is the same as the total energy in frequency domain:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |S_x(f)|^2 df = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S_x(\omega)|^2 d\omega$$

- $E(f) = |S_x(f)|^2$  - energy spectral density (ESD) of x(t).  
Represents the amount of energy per Hz of bandwidth
- Counterpart of Parseval theorem for periodic signals
- Autocorrelation property:

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t) x^*(t - \tau) dt \leftrightarrow |S_x(\omega)|^2$$

$$R_x(0) = E$$

# Parseval Theorem: Example

$$\int_{-\infty}^{+\infty} \left( \frac{\sin t}{t} \right)^2 dt = ?$$

# Fourier Transform of Real Signal

- if  $x(t)$  is real,  $\text{Im}\{x(t)\} = 0 \Rightarrow S_x(-\omega) = S_x^*(\omega)$
- Fourier transform can be presented as

$$x(t) = 2 \int_0^{\infty} |S_x(f)| \cos(2\pi f t + \varphi(f)) df,$$

$$\varphi(f) = \tan^{-1} \left( \frac{\text{Im}[S_x(f)]}{\text{Re}[S_x(f)]} \right)$$

No negative frequencies!

# Signal Bandwidth & Negative Frequencies

- What negative frequency means?
- It means that there is  $e^{-j2\pi ft}$  term in signal spectrum
- Convenient mathematical tool. Do not exist in practice (i.e., cannot be measured on spectrum analyzer)
- What is the signal bandwidth? There are many definitions.
- Defined for positive frequencies only.
- Determines the frequency band over which a substantial part of the signal power/energy is concentrated.
- For band-limited signals

$$\Delta f = f_{\max} - f_{\min}, \quad f_{\max}, f_{\min} \geq 0$$

# Power and Energy

- Power  $P_x$  & energy  $E_x$  of signal  $x(t)$  are:

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- Energy-type signals:  $E_x < \infty$
- Power-type signals:  $0 < P_x < \infty$
- Signal cannot be both energy & power type!
- Signal energy: if  $x(t)$  is voltage or current,  $E_x$  is the energy dissipated in 1 Ohm resistor
- Signal power: if  $x(t)$  is voltage or current,  $P_x$  is the power dissipated in 1 Ohm resistor.

# Energy-Type Signals (summary)

- Signal energy in time & frequency domains:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |S_x(f)|^2 df = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S_x(\omega)|^2 d\omega$$

- Energy spectral density (energy per Hz of bandwidth):

$$E_x(f) = |S_x(f)|^2$$

- ESD is FT of autocorrelation function:

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t) x^*(t - \tau) dt \leftrightarrow E_x(f)$$

$$R_x(0) = E_x$$

## Power-Type Signals: PSD

- Definition of the power spectral density (PSD) (power per Hz of bandwidth):

$$P_x(f) = \lim_{T \rightarrow \infty} \frac{|S_T(f)|^2}{T} \Rightarrow P_x = \int_{-\infty}^{\infty} P_x(f) df < \infty$$

- where  $x_T(t)$  is the truncated signal (to  $[-T/2, T/2]$ ),

$$x_T(t) = x(t) \Pi\left(\frac{t}{T}\right) = \begin{cases} x(t), & -T/2 \leq t \leq T/2 \\ 0, & \text{otherwise} \end{cases}$$

- and  $S_T(f)$  is its spectrum (FT),

$$S_T(f) = FT \{x_T(t)\}$$

# Power-Type Signals

- Time-average autocorrelation function:

$$R_x(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) x^*(t - \tau) dt$$

- Power of the signal:

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt = R_x(0)$$

- Wiener-Khintchine theorem :

$$P_x = \int_{-\infty}^{\infty} P_x(f) df \Rightarrow P_x(f) = FT \{R_x(\tau)\}$$

# Periodic Signals

- Power of a periodic signal:

$$P_x = \frac{1}{T} \int_T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = R_x(0) \quad \leftarrow x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$

- Autocorrelation function:

$$R_x(\tau) = \frac{1}{T} \int_T x(t) x^*(t - \tau) dt = \sum_{n=-\infty}^{\infty} |c_n|^2 e^{jn\omega_0 \tau}$$

- Power spectral density (PSD):

$$P_x(f) = FT\{R_x(\tau)\} = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta\left(f - \frac{n}{T}\right)$$

Prove these properties!

# Relation Between Fourier Transform & Series

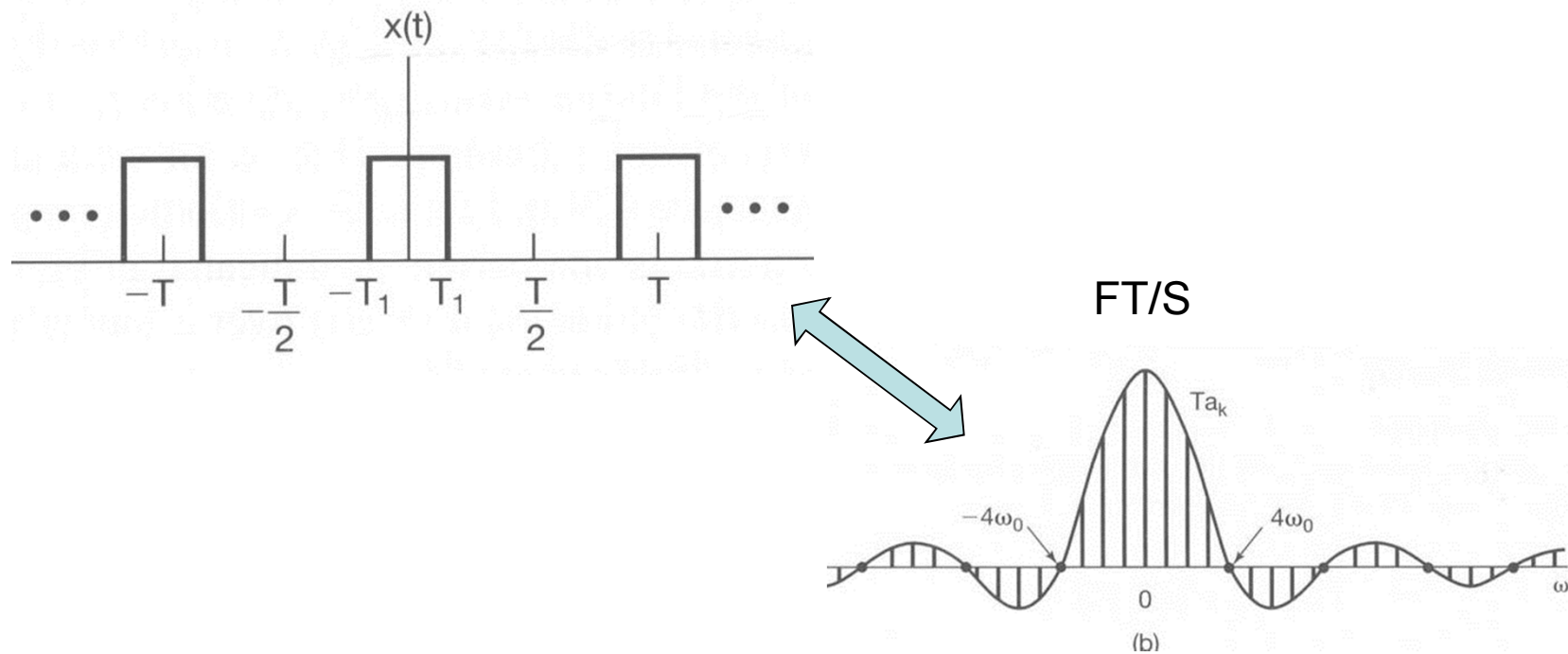
- consider a periodic signal  $x(t)=x(t+T)$
- truncate it (one period only):
 
$$x_T(t) = \begin{cases} x(t), & -T/2 < t \leq T/2 \\ 0, & \text{otherwise} \end{cases}$$

- find FT of the truncated signal  $x_T(t)$ :  $x_T(t) \leftrightarrow S_{x_T}(\omega)$
- Fourier series of the original periodic signal  $x(t)$  is

$$c_n = \frac{1}{T} S_{x_T}(n\omega_0) \quad \leftarrow \text{prove this}$$

- Continuous spectrum is the envelope of discrete spectrum (see slide 2)!

# Fourier Transform of Single Pulse ~ Envelope of Fourier Series of Pulse Train



A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.

# Summary

- Definition of Fourier Transform
- Properties of Fourier Transform
- Signal bandwidth
- Signal power & energy. Energy and power-type signals
- Fourier transform of periodic signals
- Relation between Fourier series & transform
  
- **Reading**: Couch, 2.1-2.6; Oppenheim & Willsky, Ch. 1, 3 & 4. Study carefully all the examples (including end-of-chapter study-aid examples), make sure you understand and can solve them with the book closed.
- Do some end-of-chapter problems. Students' solution manual provides solutions for many of them.