MAT 1300 Midterm Test 1

Total = 20 points

Part I. Detailed Answer Questions (2 × 4 = 8 points)

Write your answer line by line in a logical order. Use standard mathematical terminology and notations. Especially, pay attention to the use of the equal sign and the limit sign. Write the square root sign and the fraction sign long enough to cover everything it is supposed to cover. Your presentation is a part of the marking scheme. If your presentation is not satisfied, you will have your mark deducted even if your answer is correct!

Write your answer on a piece of paper, scan and convert it to a pdf or jpg file (other type of files are NOT accepted!) Use the "Add File" button under each question to upload your file to Brightspace. Upload your files as early as possible to avoid any problem you may encounter at the end of the exam. If you want to make any changes, you may upload multiple files. Just indicate in the question which file you want to be marked.

1.1. (4 points) Find derivative of function $f(x) = \dfrac{1}{1+\sqrt{x}}$ at $x = 1$ by definition.

Solution. $f(1) = 1 / 2$.

$$f'(1) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{1+\sqrt{1+h}} - \frac{1}{2}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1-\sqrt{1+h}}{2\left(1+\sqrt{1+h}\right)}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{\left(1-\sqrt{1+h}\right)\left(1+\sqrt{1+h}\right)}{2\left(1+\sqrt{1+h}\right)^2}\right)$$

$$= -\lim_{h\to 0}\frac{1}{2\left(1+\sqrt{1+h}\right)^2} = -\frac{1}{8}.$$

1.2. (4 points) Find derivative of function $y = \dfrac{1}{2+\sqrt{x}}$ at $x = 1$ by definition.

Solution. $f(1) = 1 / 3$.

$$f'(1) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{2+\sqrt{1+h}} - \frac{1}{3}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1-\sqrt{1+h}}{3\left(2+\sqrt{1+h}\right)}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{\left(1-\sqrt{1+h}\right)\left(1+\sqrt{1+h}\right)}{3\left(2+\sqrt{1+h}\right)\left(1+\sqrt{1+h}\right)}\right)$$

$$= -\lim_{h\to 0}\frac{1}{3\left(2+\sqrt{1+h}\right)\left(1+\sqrt{1+h}\right)} = -\frac{1}{18}.$$

1.3. (4 points) Find derivative of function $y = \dfrac{1}{1+\sqrt{x}}$ at $x = 4$ by definition.

Solution. $f(4) = 1/3$.

$$f'(4) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{1+\sqrt{4+h}}-\frac{1}{3}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{2-\sqrt{4+h}}{3\left(1+\sqrt{4+h}\right)}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{\left(2-\sqrt{4+h}\right)\left(2+\sqrt{4+h}\right)}{3\left(1+\sqrt{4+h}\right)\left(2+\sqrt{4+h}\right)}\right)$$

$$= -\lim_{h\to 0}\frac{1}{3\left(1+\sqrt{4+h}\right)\left(2+\sqrt{4+h}\right)} = -\frac{1}{36}.$$

1.4. (4 points) Find derivative of function $y = \dfrac{1}{2+\sqrt{x}}$ at $x = 4$ by definition.

Solution. $f(4) = 1/4$.

$$f'(4) = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{2+\sqrt{4+h}}-\frac{1}{4}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{2-\sqrt{4+h}}{4\left(2+\sqrt{4+h}\right)}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{\left(2-\sqrt{4+h}\right)\left(2+\sqrt{4+h}\right)}{4\left(2+\sqrt{4+h}\right)^2}\right)$$

$$= -\lim_{h\to 0}\frac{1}{4\left(2+\sqrt{4+h}\right)^2} = -\frac{1}{64}.$$

2.1. (4 points) Find the horizontal and vertical asymptote(s) of the function

$$f(x) = \frac{2x^2+1}{x^2-2x-3}.$$

Your answer should explain why what you find are vertical or horizontal asymptotes, and state explicitly "Function $f(x)$ has vertical (or horizontal) asymptote(s) …".

Solution. Since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) =$ 3, function $f(x)$ has horizontal asymptote $y = 2$.

$x^2 - 2x - 3 = (x + 1)(x - 3)$.

Since $\lim_{x\to 3^+} f(x) = \lim_{x\to -1^-} f(x) = \infty$, function $f(x)$ has vertical asymptotes $x = -1$ and $x = 3$.

2.2. (4 points) Find the horizontal and vertical asymptote(s) of the function

$$f(x) = \frac{3x^2+1}{x^2-4}.$$

Your answer should explain why what you find are vertical or horizontal asymptotes, and state explicitly "Function $f(x)$ has vertical (or horizontal) asymptote(s) …".

Solution. Since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 3$, function $f(x)$ has horizontal asymptote $y = 3$.

$x^2 - 4 = (x - 2)(x + 2)$.

Since $\lim_{x\to-2^-} f(x) = \lim_{x\to 2^+} f(x) = \infty$, function $f(x)$ has vertical asymptotes $x = -2$ and $x = 2$.

2.3. (4 points) Find the horizontal and vertical asymptote(s) of the function

$$f(x) = \frac{2x^2+1}{3-2x-x^2}.$$

Your answer should explain why what you find are vertical or horizontal asymptotes, and state explicitly "Function $f(x)$ has vertical (or horizontal) asymptote(s) …".

Solution. Since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = -1$, function $f(x)$ has horizontal asymptote $y = -2$.

$3 - 2x - x^2 = (3 + x)(1 - x)$.

Since $\lim_{x\to-3^+} f(x) = \lim_{x\to 1^-} f(x) = \infty$, function $f(x)$ has vertical asymptotes $x = -3$ and $x = 1$.

2.4. (4 points) Find the horizontal and vertical asymptote(s) of the function

$$f(x) = \frac{3x^2+1}{4-x^2}.$$

Your answer should explain why what you find are vertical or horizontal asymptotes, and state explicitly "Function $f(x)$ has vertical (or horizontal) asymptote(s) …".

Solution. Since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = -3$, function $f(x)$ has horizontal asymptote $y = -3$.

$4 - x^2 = (2 + x)(2 - x)$.

Since $\lim_{x\to 2^-} f(x) = \lim_{x\to -2^+} f(x) = \infty$, function $f(x)$ has vertical asymptotes $x = 2$ and $x = -2$.

Part II. Multiple-choice Questions (2 × 6 = 12 points)

3.1. Suppose some of the values of two functions $f(x)$ and $g(x)$ are given in the following table:

x	1	2	3	4	5	6
$f(x)$	2	4	5	1	6	3
$g(x)$	6	3	2	5	4	1

Then

(A) $(f \circ g)(1) = 3$ and $f^{-1}(1) = 4$;
(B) $(f \circ g)(1) = 4$ and $f^{-1}(1) = 4$;
(C) $(f \circ g)(1) = 3$ and $f^{-1}(1) = 2$;
(D) $(f \circ g)(1) = 4$ and $f^{-1}(1) = 2$;
(E) $(f \circ g)(1) = 3$ and $f^{-1}(1) = 5$;
(F) $(f \circ g)(1) = 4$ and $f^{-1}(1) = 5$.

Solution. (A) $(f \circ g)(1) = f(g(1)) = f(6) = 3$. Since $f(4) = 1$, $f^{-1}(1) = 4$.

3.2. Suppose some of the values of two functions $f(x)$ and $g(x)$ are given in the following table:

x	1	2	3	4	5	6
$f(x)$	2	4	5	1	6	3
$g(x)$	6	3	2	5	4	1

Then

(A) $(f \circ g)(2) = 5$ and $f^{-1}(2) = 1$;
(B) $(f \circ g)(2) = 1$ and $f^{-1}(2) = 1$;
(C) $(f \circ g)(2) = 5$ and $f^{-1}(2) = 4$;
(D) $(f \circ g)(2) = 1$ and $f^{-1}(2) = 4$;
(E) $(f \circ g)(2) = 5$ and $f^{-1}(2) = 6$;
(F) $(f \circ g)(2) = 1$ and $f^{-1}(2) = 6$.

Solution. (A) $(f \circ g)(2) = f(g(2)) = f(3) = 5$. Since $f(1) = 2$, $f^{-1}(2) = 1$.

3.3. Suppose some of the values of two functions $f(x)$ and $g(x)$ are given in the following table:

x	1	2	3	4	5	6
$f(x)$	2	4	5	1	6	3
$g(x)$	5	2	6	3	4	1

Then

(A) $(f \circ g)(3) = 3$ and $f^{-1}(3) = 6$;
(B) $(f \circ g)(3) = 4$ and $f^{-1}(3) = 6$;
(C) $(f \circ g)(3) = 3$ and $f^{-1}(3) = 5$;
(D) $(f \circ g)(3) = 4$ and $f^{-1}(3) = 5$;
(E) $(f \circ g)(3) = 3$ and $f^{-1}(3) = 1$;
(F) $(f \circ g)(3) = 4$ and $f^{-1}(3) = 1$.

Solution. (A) $(f \circ g)(3) = f(g(3)) = f(6) = 3$. Since $f(6) = 3$, $f^{-1}(3) = 6$.

3.4. Suppose some of the values of two functions $f(x)$ and $g(x)$ are given in the following table:

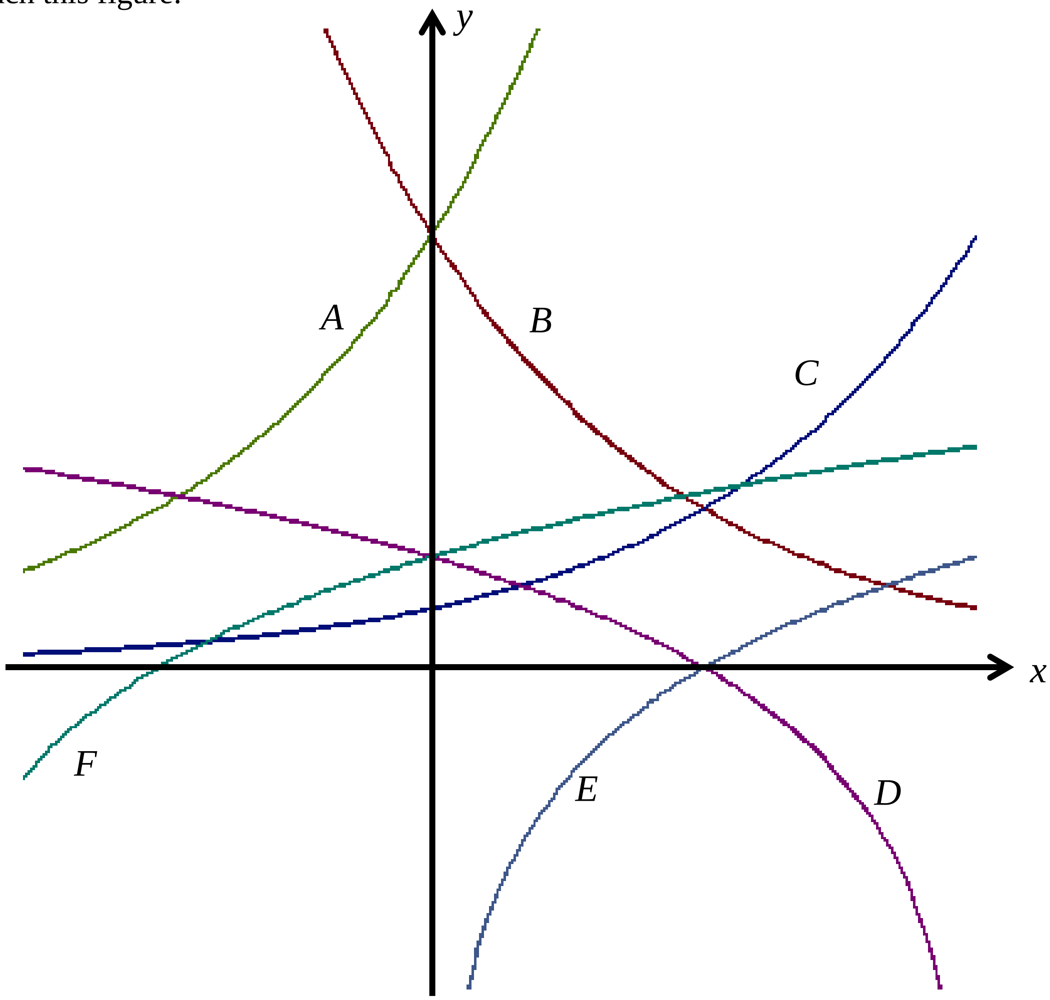
x	1	2	3	4	5	6
$f(x)$	2	4	5	1	6	3
$g(x)$	5	2	6	3	4	1

Then

(A) $(f \circ g)(6) = 2$ and $f^{-1}(6) = 5$;
(B) $(f \circ g)(6) = 6$ and $f^{-1}(6) = 5$;
(C) $(f \circ g)(6) = 2$ and $f^{-1}(6) = 3$;
(D) $(f \circ g)(6) = 6$ and $f^{-1}(6) = 3$;
(E) $(f \circ g)(6) = 2$ and $f^{-1}(6) = 4$;
(F) $(f \circ g)(6) = 6$ and $f^{-1}(6) = 4$.

Solution. (A) $(f \circ g)(6) = f(g(6)) = f(1) = 2$. Since $f(5) = 6$, $f^{-1}(6) = 5$.

4. Attach this figure:



4.1. In this figure, there are graphs of functions e^{x-1}, e^{x+1}, e^{1-x}, ln $(x + 2)$, ln x, and ln $(2 - x)$. Which one is the graph of $f(x) = e^{1-x}$?

Answer. (B) Since $e^{1-x} > 0$, (D), (E) and (F) not true. Since e^{1-x} is decreasing, the only choice is B.

4.2. In this figure, there are graphs of functions e^{x-1}, e^{1+x}, e^{1-x}, ln $(x + 2)$, ln x, and ln $(2 - x)$. Which one is the graph of $f(x) = e^{1+x}$?

Answer. (A) Since $e^{1+x} > 0$, (D), (E) and (F) not true. Since e^{1+x} is decreasing, (B) is false. Since $e^{1+x} = ee^x > e^x / e = e^{1-x}$, (A) is the right choice.

4.3. In this figure, there are graphs of functions e^{x-1}, e^{x+1}, e^{1-x}, ln $(x + 2)$, ln x, and ln $(2 - x)$. Which one is the graph of $f(x) = \ln(2 - x)$?

Answer. (D) Since, when $2 - x < 1$, (i.e., $x > 1$), $\ln(2 - x) < 0$, (D) is the only possible choice.

4.4. In this figure, there are graphs of functions e^{x-1}, e^{x+1}, e^{1-x}, $\ln(x + 2)$, $\ln x$, and $\ln(2 - x)$. Which one is the graph of $f(x) = \ln(x + 2)$?

Answer. (F) When $x + 2 < 1$, (i.e., $x < -1$), $\ln(x + 2) < 0$. The only possible choice is (F).

5.1. If function $f(x) = \begin{cases} ax^2 - 1, & \text{if } x \le 2 \\ x - a, & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$, then $a =$

(A) 2 / 5; (B) 1 / 3; (C) 1 / 2; (D) 3 / 5;
(E) 1 / 4; (F) None of the others.

Solution. (D) Let $\lim_{x\to 2^-}(ax^2 - 1) = 4a - 1 = \lim_{x\to 2^+}(x - a) = 2 - a$. Then $4a - 1 = 2 - a$, $5a = 3$, $a = 3 / 5$.

5.2. If function $f(x) = \begin{cases} ax^2 + 1, & \text{if } x \le 2 \\ x + a, & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$, then $a =$

(A) 2 / 5; (B) 1 / 3; (C) 1 / 2; (D) 3 / 5;
(E) 1 / 4; (F) None of the others.

Solution. (B) Let $\lim_{x\to 2^-}(ax^2 + 1) = 4a + 1 = \lim_{x\to 2^+}(x + a) = 2 + a$. Then $4a + 1 = 2 + a$, $3a = 1$, $a = 1 / 3$.

5.3. If function $f(x) = \begin{cases} ax^2 - 1, & \text{if } x \le 3 \\ x - a, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$, then $a =$

(A) 2 / 5; (B) 1 / 3; (C) 1 / 2; (D) 3 / 5;
(E) 1 / 4; (F) None of the others.

Solution. (A) Let $\lim_{x\to 3^-}(ax^2 - 1) = 9a - 1 = \lim_{x\to 3^+}(x - a) = 3 - a$. Then $9a - 1 = 3 - a$, $10a = 4$, $a = 2 / 5$.

5.4. If function $f(x) = \begin{cases} ax^2 + 1, & \text{if } x \le 3 \\ x + a, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$, then $a =$

(A) 2 / 5; (B) 1 / 3; (C) 1 / 2; (D) 3 / 5;
(E) 1 / 4; (F) None of the others.

Solution. (E) Let $\lim_{x\to 3^-}(ax^2+1) = 9a+1 = \lim_{x\to 3^+}(x+a) = 3+a$. Then 9*a* + 1 = 3 + *a*, 8*a* = 2, *a* = 1 / 4.

6.1. At which value of *x*, does the graph of function *y* = $xe^{1/\sqrt{x}}$ have a horizontal tangent line?

(A) 1 / 4; (B) 1 / 2; (C) 1; (D) 2; (E) 3; (F) 4.

Solution. (A) $f'(x) = e^{1/\sqrt{x}} + xe^{1/\sqrt{x}}\left(-\frac{1}{2x^{3/2}}\right) = \frac{e^{1/\sqrt{x}}}{2\sqrt{x}}(2\sqrt{x}-1)$. Let $f'(x) = 0$. $2\sqrt{x} - 1 = 0$, *x* = 1 / 4.

6.2. At which value of *x*, does the graph of function *y* = $\sqrt{x}e^{-x/2}$ have a horizontal tangent line?

(A) 1 / 4; (B) 1 / 2; (C) 1; (D) 2; (E) 3; (F) 4.

Solution. (C) $f'(x) = \frac{1}{2\sqrt{x}}e^{-x/2} + \sqrt{x}e^{-x/2}\left(-\frac{1}{2}\right) = \frac{e^{-x}}{2\sqrt{x}}(1-x)$. Let $f'(x) = 0$. Then *x* = 1.

6.3. At which value of *x*, does the graph of function *y* = $\sqrt{x}e^{1/x}$ have a horizontal tangent line?

(A) 1 / 4; (B) 1 / 2; (C) 1; (D) 2; (E) 3; (F) 4.

Solution. (D) $f'(x) = \frac{1}{2\sqrt{x}}e^{1/x} + \sqrt{x}e^{1/x}\left(-\frac{1}{x^2}\right) = \frac{1}{2\sqrt{x}}e^{1/x} - \frac{e^{1/x}}{x^{3/2}} = \frac{1}{2x^{3/2}}e^{1/x}(x-2)$. Let $f'(x) = 0$. Then *x* = 2.

6.4. At which value of *x*, does the graph of function *y* = $xe^{-\sqrt{x}}$ have a horizontal tangent line?

(A) 1 / 4; (B) 1 / 2; (C) 1; (D) 2; (E) 3; (F) 4.

Solution. (F) $f'(x) = e^{-\sqrt{x}} + xe^{-\sqrt{x}}\left(-\frac{1}{2\sqrt{x}}\right) = e^{-\sqrt{x}}\left(1-\frac{1}{2}\sqrt{x}\right)$. Let $f'(x) = 0$. $\sqrt{x} = 2$, *x* = 4.

7.1. Let $f(x) = \dfrac{\ln(x+1)}{2x+1}$. Then $f'(1) =$

(A) $\frac{1}{18}(3-4\ln 2)$; (B) $\frac{2}{9}(3-2\ln 2)$; (C) $2 - 4 \ln 2$;

(D) $\frac{1}{8}(2-\ln 2)$; (E) $\frac{1}{3}(4-3\ln 2)$; (F) $\frac{1}{6}(3-\ln 2)$.

Solution. $f'(x) = \dfrac{\frac{2x+1}{x+1} - 2\ln(x+1)}{(2x+1)^2} = \dfrac{(2x+1)-2(x+1)\ln(x+1)}{(2x+1)^2(x+1)}$. $f'(1) = \dfrac{3-4\ln 2}{18}$.

7.2. Let $f(x) = \dfrac{\ln(2x+1)}{x+1}$. Then $f'\left(\frac{1}{2}\right) =$

(A) $\frac{1}{18}(3-4\ln 2)$; (B) $\frac{2}{9}(3-2\ln 2)$; (C) $2 - 4 \ln 2$;

(D) $\frac{1}{8}(2-\ln 2)$; (E) $\frac{1}{3}(4-3\ln 2)$; (F) $\frac{1}{6}(3-\ln 2)$.

Solution. $f'(x) = \dfrac{\frac{2(x+1)}{2x+1} - \ln(2x+1)}{(x+1)^2} = \dfrac{2(x+1)-(2x+1)\ln(2x+1)}{(x+1)^2(2x+1)}$. $f'\left(\frac{1}{2}\right) = \frac{2}{9}(3-2\ln 2)$.

7.3. Let $f(x) = \dfrac{\ln(2x-1)}{x-1}$. Then $f'\left(\frac{3}{2}\right) =$

(A) $\frac{1}{18}(3-4\ln 2)$; (B) $\frac{2}{9}(3-2\ln 2)$; (C) $2 - 4 \ln 2$;

(D) $\frac{1}{8}(2-\ln 2)$; (E) $\frac{1}{3}(4-3\ln 2)$; (F) $\frac{1}{6}(3-\ln 2)$.

Solution. $f'(x) = \dfrac{\frac{2(x-1)}{2x-1} - \ln(2x-1)}{(x-1)^2} = \dfrac{2(x-1)-(2x-1)\ln(2x-1)}{(x-1)^2(2x-1)}$. $f'\left(\frac{3}{2}\right) = 2(1 - 2\ln 2)$.

7.4. Let $f(x) = \dfrac{\ln(2x-1)}{2x+1}$. Then $f'\left(\frac{3}{2}\right) =$

(A) $\frac{1}{18}(3-4\ln 2)$; (B) $\frac{2}{9}(3-2\ln 2)$; (C) $2 - 4\ln 2$;
(D) $\frac{1}{8}(2-\ln 2)$; (E) $\frac{1}{3}(4-3\ln 2)$; (F) $\frac{1}{6}(3-\ln 2)$.

Solution. $f'(x) = \dfrac{\frac{2(2x+1)}{2x-1} - 2\ln(2x-1)}{(2x+1)^2} = \dfrac{2(2x+1)-2(2x-1)\ln(2x-1)}{(2x+1)^2(2x-1)}$. $f'\left(\frac{3}{2}\right) = \frac{1}{8}(2-\ln 2)$.

8.1. The graph of function $f(x) = 6x^5 - 5x^4$ is concave up in interval(s)

(A) $(0.5, \infty)$ only; (B) $(-\infty, 0)$ and $(0.5, \infty)$ only; (C) $(-\infty, 0)$ only;
(D) $(0, 0.5)$ only; (E) $(-\infty, 0.5)$ only; (F) $(0, \infty)$ only.

Solution. (A) $f'(x) = 30x^4 - 20x^3 = 10(3x^4 - 2x^3)$. $f''(x) = 10(12x^3 - 6x^2) = 60x^2(2x - 1)$. $f''(x) > 0$ if and only if $2x - 1 > 0$, or equivalently, $x > 0.5$.

8.2. The graph of function $f(x) = 6x^5 - 10x^4 + 5x^3$ is concave down in interval(s)

(A) $(0.5, \infty)$ only; (B) $(-\infty, 0)$ and $(0.5, \infty)$ only; (C) $(-\infty, 0)$ only;
(D) $(0, 0.5)$ only; (E) $(-\infty, 0.5)$ only; (F) $(0, \infty)$ only.

Solution. (C) $f'(x) = 30x^4 - 40x^3 + 15x^2 = 5(6x^4 - 8x^3 + 3x^2)$. $f''(x) = 5(24x^3 - 24x^2 + 6x) = 30x(4x^2 - 4x + 1) = 30x(2x - 1)^2$. $f''(x) < 0$ if and only if $x < 0$.

8.3. The graph of function $f(x) = 6x^5 + 5x^4$ is concave down in interval(s)

(A) $(0, \infty)$ only; (B) $(-\infty, -0.5)$ and $(0, \infty)$ only; (C) $(-\infty, -0.5)$ only;
(D) $(-0.5, 0)$ only; (E) $(-\infty, 0)$ only; (F) $(-0.5, \infty)$ only.

Solution. (C) $f'(x) = 30x^4 + 20x^3 = 10(3x^4 + 2x^3)$. $f''(x) = 10(12x^3 + 6x^2) = 60x^2(2x + 1)$. $f''(x) < 0$ if and only if $2x + 1 < 0$, or equivalently, $x < -0.5$.

8.4. The graph of function $f(x) = 6x^5 + 10x^4 + 5x^3$ is concave up in interval(s)

(A) $(0, \infty)$ only; (B) $(-\infty, -0.5)$ and $(0, \infty)$ only; (C) $(-\infty, -0.5)$ only;
(D) $(-0.5, 0)$ only; (E) $(-\infty, 0)$ only; (F) $(-0.5, \infty)$ only.

Solution. (A) $f\,'(x) = 30x^4 + 40x^3 + 15x^2 = 5(6x^4 + 8x^3 + 3x^2)$. $f\,''(x) = 5(24x^3 + 24x^2 + 6x) = 30x(4x^2 + 4x + 1) = 30x(2x + 1)^2$. $f\,''(x) > 0$ if and only if $x > 0$.