Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

SESSION: WINTER 2022 INSTRUCTOR: HAI YAN LIU(JACK)

DISCRETE MATHEMATICS FOR COMPUTING

MAT1348 — Test 1 — Monday, February 7, 2022

- Clearly write your name and student number on this exam, and **sign it** below to confirm that you will read and follow these **instructions**:
- This is a **70+10-minute closed-book** exam. **No notes.**
- One (1) page both-sides, hand-written, (non-mechanically reproduced), 8.5x11 (Letter-size) sheet
- This exam consists of 7 questions on 9 pages (including this cover page). The total number of points possible is **28 points**.
- Questions 1–3 are **multiple-choice** worth **8 points** total. In each question, you must select the correct response. You do not need to justify your answers.
- Questions 4–7 are **long-answer** worth **20 points** total. To receive full marks, your solution/proof must be complete, correct, and show all relevant details.
- On page 9, there is a **Table of Logical Equivalences.** You may detach page 9.
- Read all questions carefully and be sure to follow the instructions for the individual problems.
- You must use **proper mathematical notation and terminology**.
- For rough work or additional space, you may use the backs of pages.

Cellular phones, unauthorized electronic devices, or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this exam.

† **By signing below, you acknowledge that you have read, understand, and will comply with the above instructions.**

FAMILY NAME: STUDENT NUMBER:

FIRST NAME: †SIGNATURE:

Good luck!

Table of Points for marking purposes. *Do not write in this table.*

	multiple-choice	Q4	Q5	Q6	Q7	TOTAL
max points possible	8 pts	5 pts	5 pts	5 pts	5 pts	28 points
points obtained						

Multiple-Choice Questions.

Write your choices in the answer boxes. No justification is needed.

Q1. Here is the truth table for 4 *mystery* compound propositions P_1, P_2, P_3, and C each consisting of atomic variables x, y, and z:

x	y	z	P_1	P_2	P_3	C	$\neg z$	$P_1 \oplus P_2$	$P_3 \oplus C$	$(P_1 \oplus P_2) \wedge (P_3 \oplus C)$
T	**T**	**T**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**
T	**T**	**F**	**T**	**T**	**F**	**T**	**T**	**F**	**T**	**F**
T	**F**	**T**	**F**	**F**	**T**	**F**	**F**	**F**	**T**	**F**
T	**F**	**F**	**T**	**T**	**F**	**F**	**T**	**F**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**F**	**F**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**	**F**	**T**	**T**	**F**	**T**	**F**
F	**F**	**T**	**F**	**T**	**F**	**T**	**F**	**T**	**T**	**T**
F	**F**	**F**	**T**	**T**	**F**	**F**	**T**	**F**	**F**	**F**

Which 4 of the following statements are true? *Only 4 statements are true.*

A. The set $\{P_1, P_2\}$ is **consistent**. True

B. The set $\{P_1, P_2, P_3\}$ is **consistent**. False

C. The argument $(P_1 \wedge P_2) \rightarrow C$ is **valid**. false

D. The argument $(P_1 \wedge P_2 \wedge P_3) \rightarrow C$ is **valid**. True

E. $(P_1 \oplus P_2) \wedge (P_3 \oplus C)$ is a **tautology**. false

F. $(x \vee \neg y \vee z) \wedge (\neg x \vee y \vee z)$ is a **disjunctive normal form (DNF)** for P_3. false, not DNF

G. $(P_1 \oplus P_2) \wedge (P_3 \oplus C)$ is a **contingency**. true

H. $(P_1 \oplus P_2) \wedge (P_3 \oplus C)$ is a **contradiction**. false

I. $\neg z$ is a **disjunctive normal form (DNF)** for P_1. true but tricky

A, D, G, I

Answers: ☐ ☐ ☐ ☐ **[4 points]**

Q2. Consider the following propositional variables:

W: “Mark writes a deferred exam.”
M: “ Mark missed the final exam accidently.”
G: “Mark gets an A+ in this course.”

Now consider the following compound proposition:

> “(A necessary condition for Mark to get an A+ in this course is that Mark does not miss the final exam accidently) unless Mark does not write a deferred exam.”

Which of the following is a correct translation of the above compound proposition?

A. $(G \to \neg M) \lor \neg W$ True **B.** $(G \to \neg M) \land \neg W$ **C.** $(\neg M \to G) \land \neg W$

D. $(\neg G \to M) \lor \neg W$ **E.** $(\neg G \to \neg M) \lor \neg W$ **F.** $(\neg M \to G) \lor \neg W$

G. none of the above

Answer is A

Answer: ☐ **[2 points]**

Q3. Let p, q and r be propositional variables. Which one the following is logically equivalent to propositional logical formula $(p \lor r) \to (q \leftrightarrow \neg r)$ circle the best response. **[2 points]**

A. $(\neg p \land \neg r) \lor (q \land \neg r) \lor (\neg q \land r)$. true

B. $(p \land r) \lor (q \land \neg r) \lor (\neg q \land r)$.

C. $(\neg p \land \neg r) \lor (\neg q \land \neg r) \lor (q \land r)$.

D. $(p \land \neg r) \lor (\neg q \land r) \lor (q \land r)$.

E. $(\neg p \land r) \lor (q \land \neg r) \lor (\neg q \land \neg r)$.

F. $(p \land r) \lor (q \land r) \lor (\neg q \land r)$.

Answer : A : $(p \lor r) \to (q \leftrightarrow \neg r) = \neg(p \lor r) \lor (q \leftrightarrow \neg r) = (\neg p \land \neg r) \lor (q \land \neg r) \lor (\neg q \land r)$

Answer: ☐ **[2 points]**

Long-Answer Questions.

Detailed justifications are required.

Q4. For this question, you will prove logical equivalence in two ways. **[5 points]**

4a. Using the laws from the Table of Equivalences given on page 9, prove that:

$$(a \vee b) \wedge ((\neg c \to \neg a) \wedge (\neg c \to \neg b)) \equiv (a \vee b) \wedge c$$

You must use **one and only one law per step**, and name the law you are using at each step.

$$\begin{aligned}
(a \vee b) \wedge ((\neg c \to \neg a) \wedge (\neg c \to \neg b)) &\equiv (a \vee b) \wedge [(\neg\neg c \vee \neg a) \wedge (\neg\neg c \vee \neg b)] \text{ implication law} \\
&\equiv (a \vee b) \wedge [(c \vee \neg a) \wedge (c \vee \neg b)]\text{Doble negation} \\
&\equiv (a \vee b) \wedge [c \vee (\neg a \wedge \neg b)] \text{ Distribution law} \\
&\equiv (a \vee b) \wedge [c \vee (\neg(a \vee b))] \text{ Demorgan's Law} \\
&\equiv ((a \vee b) \wedge c) \vee [(a \vee b) \wedge (\neg(a \vee b))]\text{Distribution law} \\
&\equiv ((a \vee b) \wedge c) \vee F \text{ neagtion law} \\
&\equiv ((a \vee b) \wedge c) \text{ Identity law}
\end{aligned}$$

4b. Now, prove that $(a \vee b) \wedge ((\neg c \to \neg a) \wedge (\neg c \to \neg b))$ is **logically equivalent to** $(a \vee b) \wedge c$ using a truth **table** and a brief explanation.

a	b	c	$\neg c \to \neg a$	$\neg c \to \neg b$	$(\neg c \to \neg a) \wedge (\neg c \to \neg b)$	X	$(a \vee b) \wedge c$
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**F**	**F**	**F**	**F**	**F**
T	**F**	**T**	**T**	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**	**F**	**F**
F	**T**	**T**	**T**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**F**	**F**	**F**	**F**
F	**F**	**T**	**T**	**T**	**T**	**F**	**F**
F	**F**	**F**	**T**	**T**	**T**	**F**	**F**

$X = (a \vee b) \wedge ((\neg c \to \neg a) \wedge (\neg c \to \neg b))$.

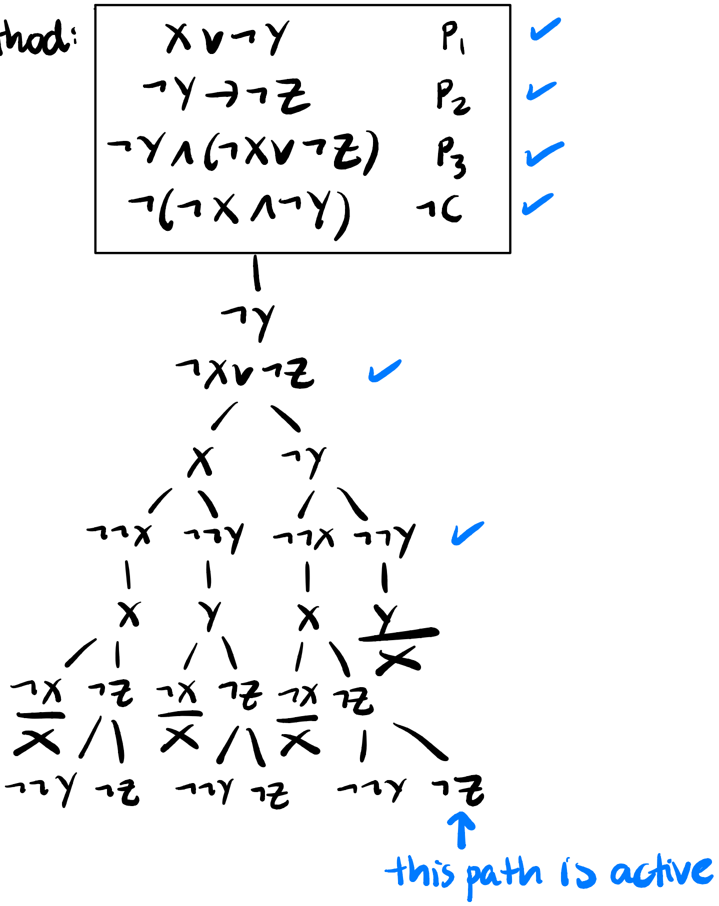
Q5. **[5 points]**

Use the method of your choice to determine whether or not the following argument is **valid**. Show your work in the space below.

Argument:

P_1 $X \vee \neg Y$
P_2 $\neg Y \to \neg Z$
P_3 $\neg Y \wedge (\neg X \vee \neg Z)$
C $\therefore\ \neg X \wedge \neg Y$

Truth tree method:



Is the argument valid? *Circle:* YES [✓ NO]

If you circle **YES**, briefly explain, referring to your work above.

If you circle **NO**, give **all counterexamples** and briefly explain.

Counter examples: When $X = T, Y = F, Z = F$, both premises are True, but conclusion is False.

X	Y	Z	$X \vee \neg Y$	$\neg Y \to \neg Z$	$\neg Y \wedge (\neg X \vee \neg Z)$	$\neg X \wedge \neg Y$
T	**T**	**T**	**T**	**T**	**F**	**F**
T	**T**	**F**	**T**	**T**	**F**	**F**
T	**F**	**T**	**T**	**F**	**F**	**F**
T	**F**	**F**	**T**	**T**	**T**	**F**
F	**T**	**T**	**F**	**T**	**F**	**F**
F	**T**	**F**	**F**	**T**	**F**	**F**
F	**F**	**T**	**T**	**T**	**T**	**T**
F	**F**	**F**	**T**	**T**	**T**	**T**

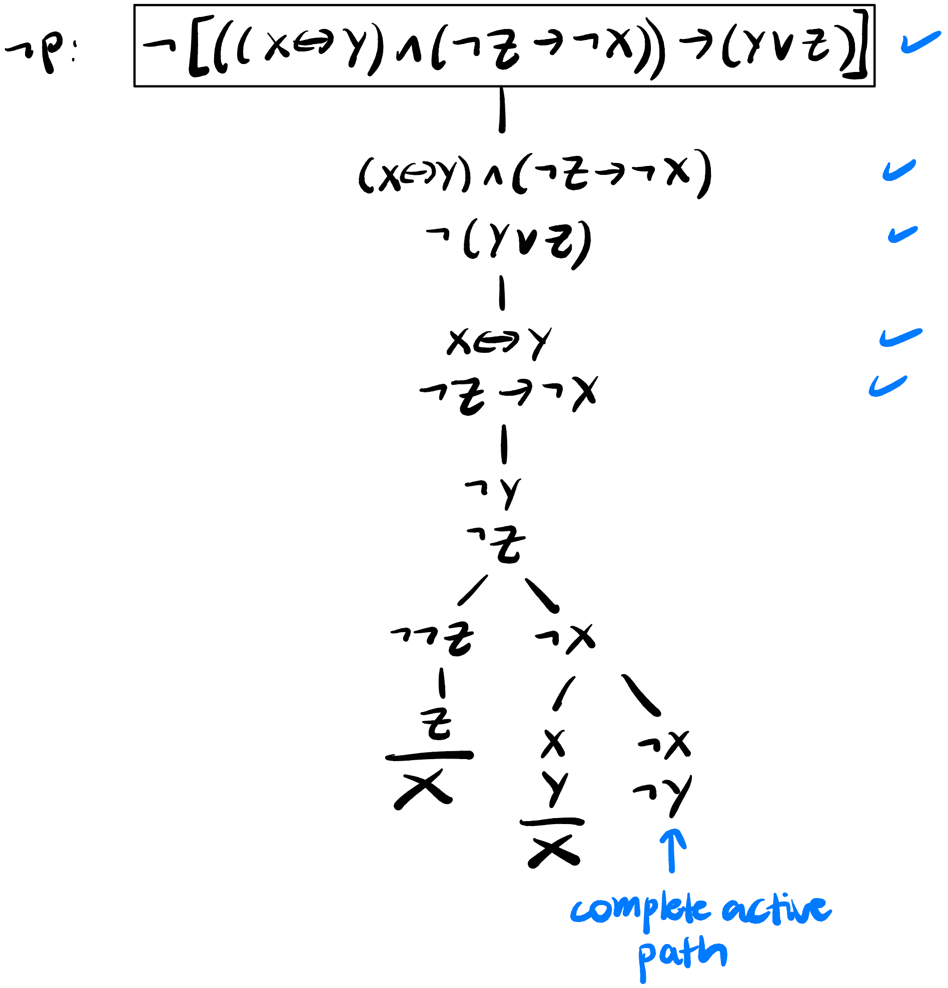
Counter examples: When $X = T, Y = F, Z = F$, both premises are True , but conclusion is False.

Q6. Consider the compound proposition P, defined as follows: **[5 points]**

$$P : \qquad \Big[\Big((X \leftrightarrow Y) \wedge (\neg Z \rightarrow \neg X)\Big) \rightarrow \Big(Y \vee Z\Big)\Big]$$

Use an appropriate **truth tree** to determine whether P is a **tautology.** Grow a complete truth tree. Make sure you apply the *official* branching rules to the propositions as they are written (*i.e. do not use logical equivalences to change the propositions in your tree – stick to the official branching rules*).

Complete truth tree: What is the root of your tree? *Circle:* P [✓ $\neg P$]



Is P tautology ? *Circle:* **YES** [✓ **NO**]

Explain your answer making reference to your tree, its root, and any relevant paths in the tree, and give all counterexamples if applicable.

In $\neg P$ tree, there is at least one complete active path which tells us that the root $\neg P$ can be true . So P itself can be false. So P is not a tautology . $\neg P$ is T, when $X = F, Y = F, Z = F$, so P is F when $X = F, Y = F, Z = F$.

Q7. For this question, you will give an indirect proof of the following Theorem 1: [**5 points**]

Let a is a rational and r is an irrational, then $x = a + r$ is an irrational.

(a) Start by writing the **contrapositive** of this statement (in English):

If $x = a + r$ is not an irrational number, then a is not a rational or r is not an irrational

(b) Complete the definition : A real number a is a rational number if :

$x = \frac{m}{n}$ for some integers ,m, n such that $n \neq 0$.

(c) Give an **Indirect Proof** of the following theorem:

Theorem 1. Let a be a rational, and r is an irrational, then $x = a + r$ is an irrational .

IMPORTANT! *In each step of your proof, you must clearly indicate whether you are assuming something, or whether what you wrote is something that follows from a definition or a previous step of your proof. If any variables appear in your proof, make sure you clearly write what they represent.*

Indirect proof of Theorem 1.
We use contradiction $P \wedge \neg Q$.
Let $a = \frac{p}{q}$, where p, q are integers , $q \neq 0$
Assume $\neg Q$ is true, $x = a + r$ is a rational number . Then x can be expressed as a fraction $x = \frac{m}{n}$
Now $x = a + r = \frac{p}{q} + r = \frac{m}{n}$. Hence
$r = \frac{m}{n} - \frac{p}{q} = \frac{mq-np}{nq}$. Since $mq - np$ and nq are integers, so r is a rational, this is contradiction with r is rational . So $P \rightarrow Q$ is true.

	Equivalence	Name
1.	$P \to Q \equiv \neg P \vee Q$	Implication Law
2. 3.	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$ $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$	Biconditional Laws
4. 5.	$P \vee \neg P \equiv \mathbf{T}$ $P \wedge \neg P \equiv \mathbf{F}$	Negation Laws
6. 7.	$P \vee \mathbf{F} \equiv P$ $P \wedge \mathbf{T} \equiv P$	Identity Laws
8. 9.	$P \vee \mathbf{T} \equiv \mathbf{T}$ $P \wedge \mathbf{F} \equiv \mathbf{F}$	Domination Laws
10. 11.	$P \vee P \equiv P$ $P \wedge P \equiv P$	Idempotent Laws
12.	$\neg\neg P \equiv P$	Double Negation Law
13. 14.	$P \vee Q \equiv Q \vee P$ $P \wedge Q \equiv Q \wedge P$	Commutative Laws
15. 16.	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	Associative Laws
17. 18.	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	Distributive Laws
19. 20.	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	De Morgan's Laws
21. 22.	$P \vee (P \wedge Q) \equiv P$ $P \wedge (P \vee Q) \equiv P$	Absorption Laws

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