MAT 1320 E

Fall 2021

Lecture 1

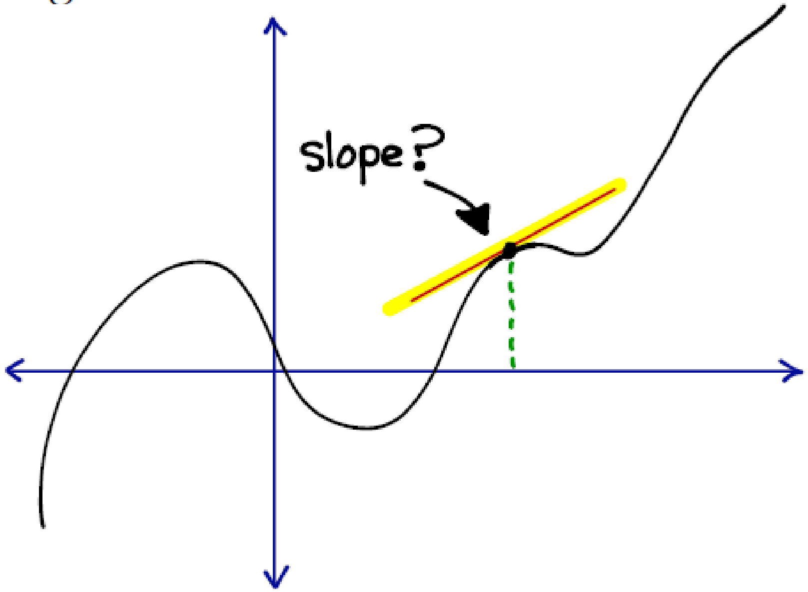
How it all started

Calculus was developed in the mid seventeenth century at about the same time by two scientists: Sir Isaac Newton and Gottfried Leibniz.

The idea was to develop general solutions to two problems posed by the Greeks:

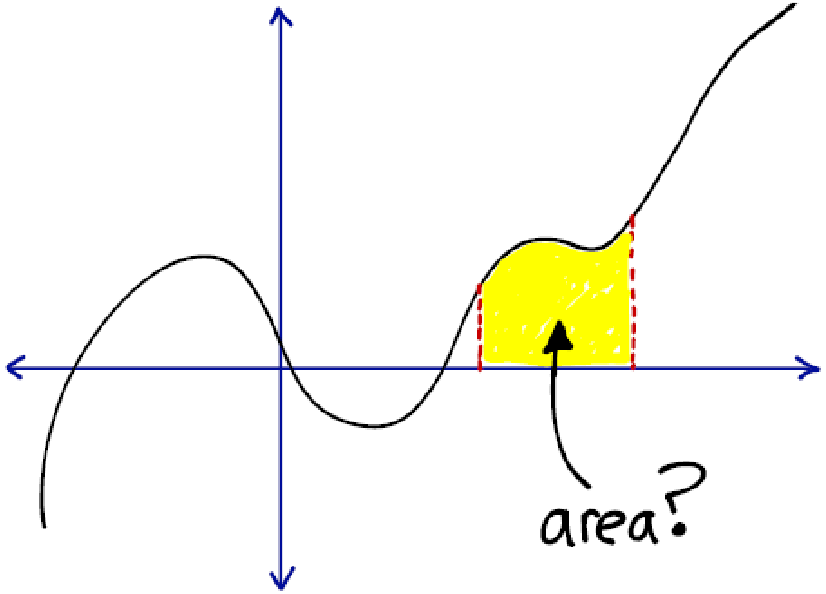
Differential Calculus
(about 2/3 of MAT1320)

Goal: find the slope of a tangent at any point on a given function



Integral Calculus
(about 1/3 of MAT1320)

Goal: find the area bounded by a function and the x-axis over a given interval

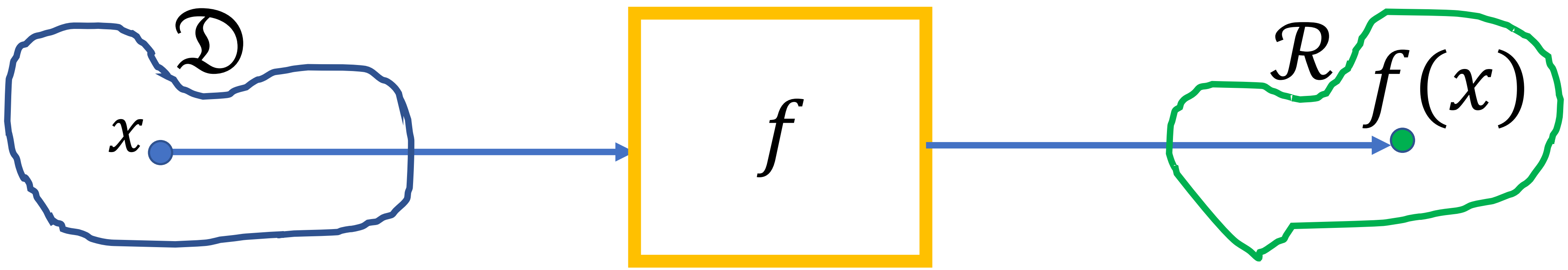


Functions and Concepts

Definition: A **function** f is a rule between two sets, called the **domain** and the **range**, that assigns to each element x in the domain **exactly one** element, called $f(x)$, in the range.

When we write $y = f(x)$, we are saying that:

1. x is the independent variable, which is an element of the domain $\mathfrak{D} \subseteq \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers
2. $f(x)$ is the value of function f at x.
3. y is the dependent variable, which is an element of the range $\mathcal{R} \subseteq \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers



Question: Which of the following relations are functions (why? Or why not?)

A. $y = \pi x^2$
B. $y^2 = 4x$
C. $y = \sqrt{|x|}$
D. $x^2 + y^2 = 4$

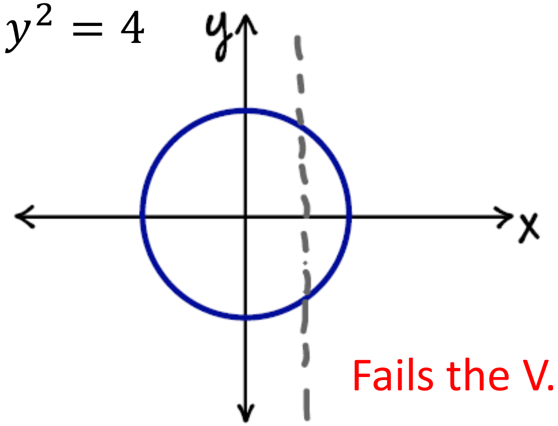
Answer: A and C are functions, since they satisfy the definition of a function.
B and D are not functions, since for each value of x there are two corresponding values of y.

The **graph** of $y = f(x)$ consists of all ordered pairs (coordinates) $(x, f(x))$ such that x belongs to the domain of f.

Vertical Line Test: (V.L.T.)

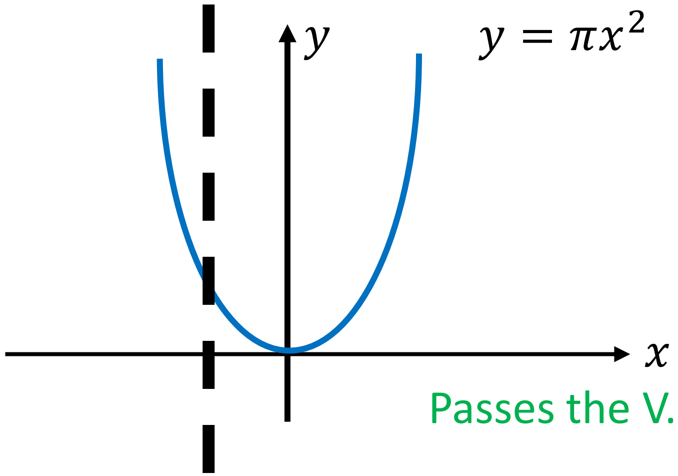
Actually, any equation in 2 variables can be represented by some curve in the xy-plane, but not every 2-dimensional curve corresponds to the graph of a **function!** The graph of a function must pass the Vertical Line Test.

$x^2 + y^2 = 4$



Fails the V.L.T.

$y = \pi x^2$

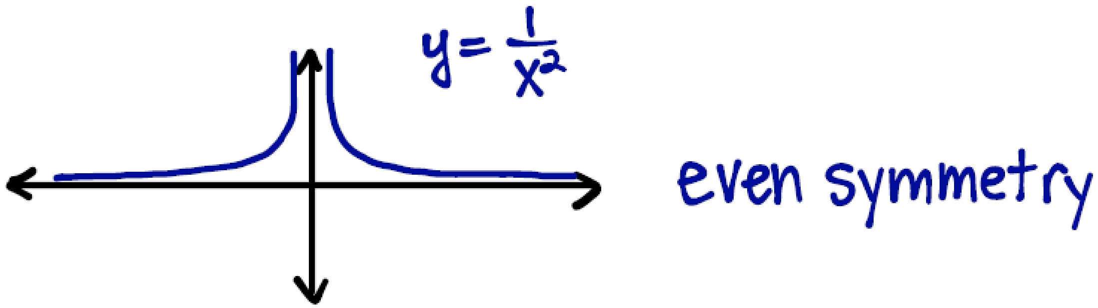


Passes the V.L.T.

Even Functions: If $f(-x) = f(x)$ for every x in the domain of f, then f is called an **even** function.

Ex. $f(x) = \frac{1}{x^2}$ is even.

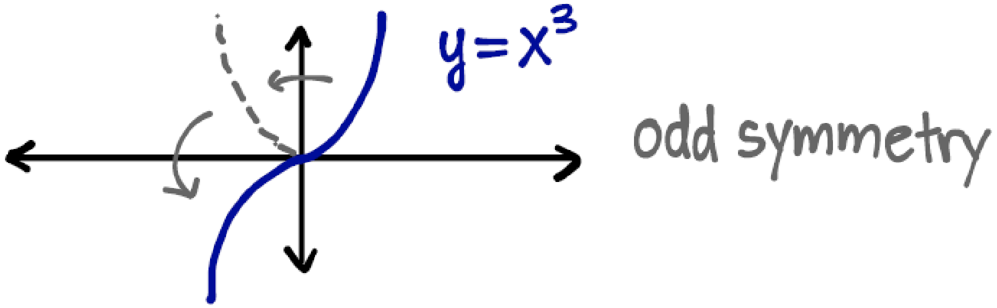
$f(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = f(x)$



Odd Functions: If $f(-x) = -f(x)$ for every x in the domain of f, then f is called an **odd** function.

Ex. $g(x) = x^3$ is odd.

$g(-x) = (-x)^3 = -x^3 = -g(x)$

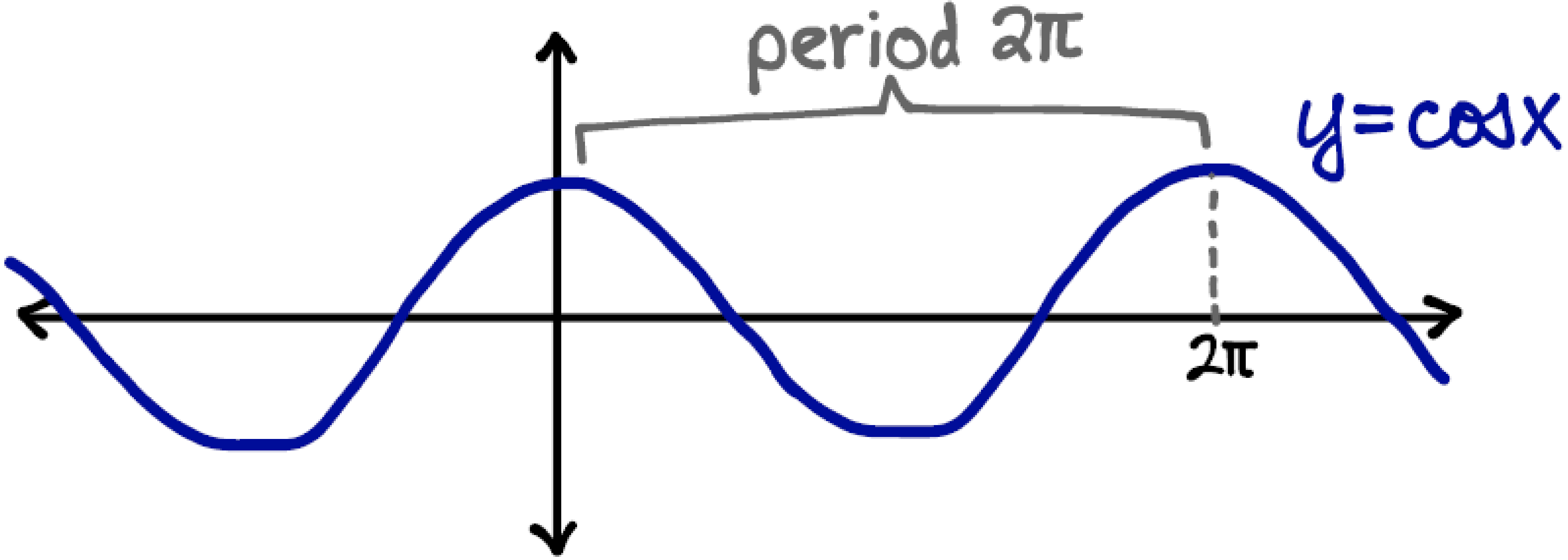


Periodic Functions: If there exists a positive constant p such that $f(x+p) = f(x)$ for every x in the domain of f, then f is called a **periodic** function. The smallest such constant p is called the **period**.

Ex $y = \cos x$ is periodic

$\cos(x+2\pi) = \cos(x)$

for all $x \in \mathbb{R}$



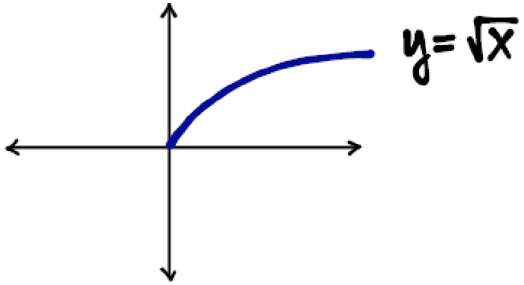
Transformations

Let $y = f(x)$ be a function and let k be a positive real number.

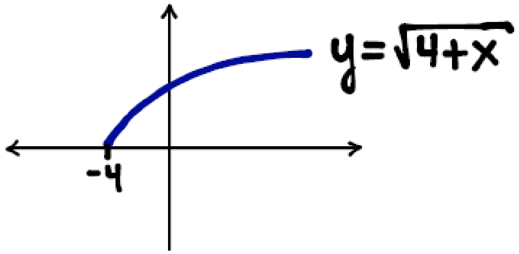
Horizontal Shifts:	• The graph of $f(x-k)$ shifts the graph of $f(x)$ to the right k units. • The graph of $f(x+k)$ shifts the graph of $f(x)$ to the left k units.
Vertical Shifts:	• The graph of $f(x)+k$ shifts the graph of $f(x)$ up k units. • The graph of $f(x)-k$ shifts the graph of $f(x)$ down k units.
Reflections:	• The graph of $-f(x)$ reflects the graph of $f(x)$ about the x-axis. • The graph of $f(-x)$ reflects the graph of $f(x)$ about the y-axis.
Vertical Scaling:	• The graph of $kf(x)$ stretches the graph of $f(x)$ vertically by a factor of k. **Note.** Stretching vertically by a factor of k is the same as compressing vertically by a factor of $\frac{1}{k}$. Think about what it means if $0 < k < 1$.
Horizontal Scaling:	• The graph of $f(kx)$ compresses the graph of $f(x)$ horizontally by a factor of k. **Note.** Compressing horizontally by a factor of k is the same as stretching horizontally by a factor of $\frac{1}{k}$. Think about what it means if $0 < k < 1$.

Example 1.2. Sketch the graph of $f(x) = 3 + \sqrt{4-x}$ using transformations; start from $y = \sqrt{x}$.

$\sqrt{x}$

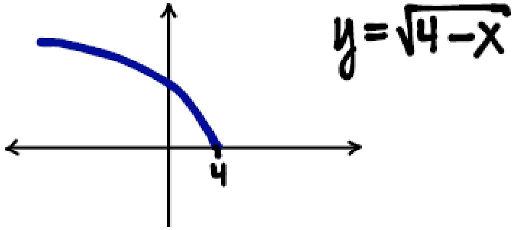


$\sqrt{4+x}$



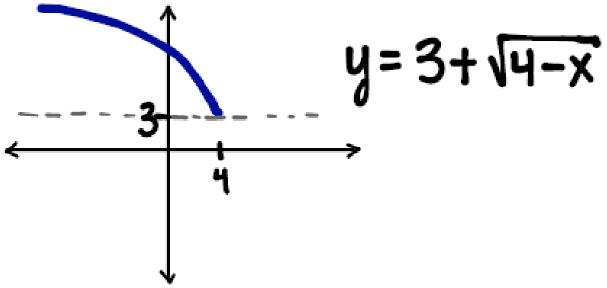
$\boxed{\sqrt{4+x} \text{ is a horizontal shift of } \sqrt{x} \text{ 4 units left}}$

$\sqrt{4+(-x)}$



$\boxed{\sqrt{4+(-x)} \text{ is a reflection of } \sqrt{4+x} \text{ about y-axis}}$

$3+\sqrt{4+(-x)}$



$\boxed{3+\sqrt{4-x} \text{ is a vertical shift of } \sqrt{4-x} \text{ 3 units up}}$

Increasing / Decreasing

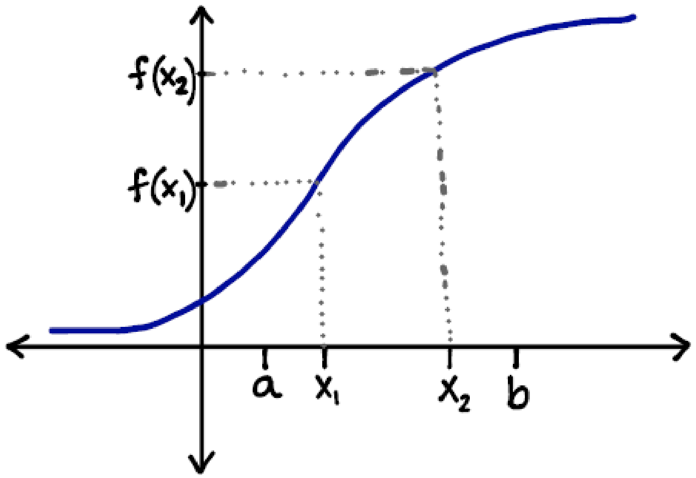
Let I be some interval of the real number line.

A function $y = f(x)$ is called...

- **increasing** on the interval I if

  $f(x_1) < f(x_2)$

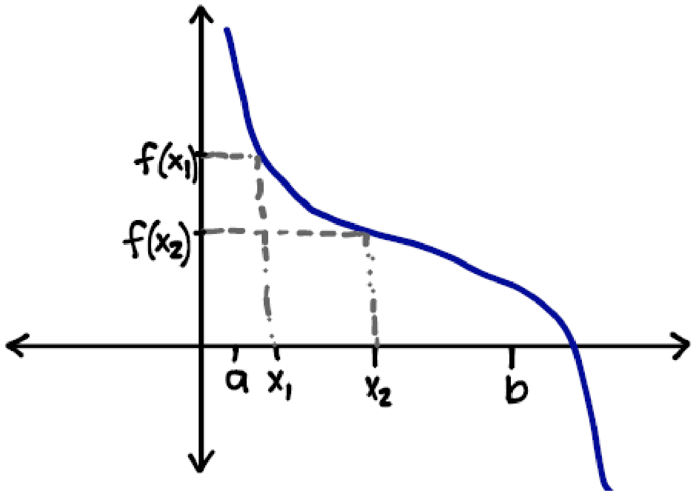
  whenever $x_1 < x_2$ and $x_1, x_2 \in I$



- **decreasing** on the interval I if

  $f(x_1) > f(x_2)$

  whenever $x_1 < x_2$ and $x_1, x_2 \in I$



Ex. $I = [a, b]$

Catalogue of Important Functions: Linear Functions

Linear Functions: $y = mx + b$

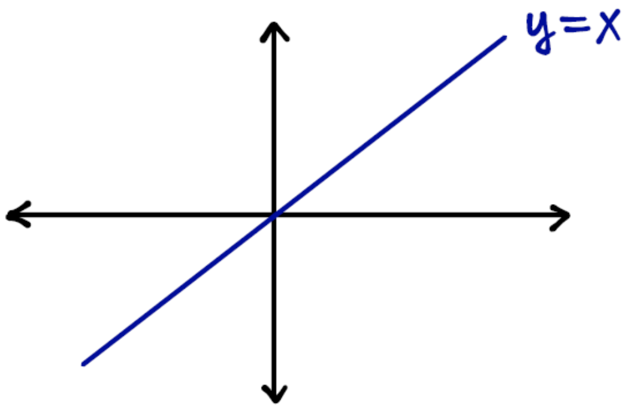
Slope $m = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$

y-intercept @ $(0, b)$

domain: $\mathbb{R}$

range:
- $\mathbb{R}$ if $m \neq 0$
- $\{b\}$ if $m = 0$

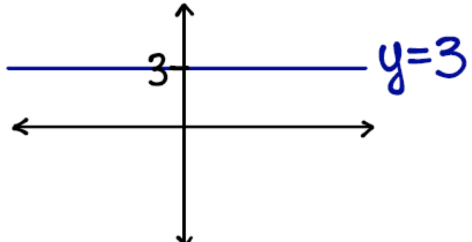
Ex



Horizontal Line. $y = b$ (constant function)

- Slope is $m = 0$
- domain: $\mathbb{R}$
- range: $\{b\}$

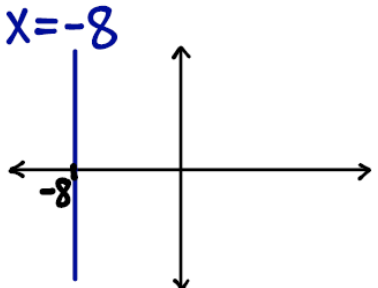
Ex



Vertical Line. $x = a$ ← x-intercept

- Slope is <u>undefined</u>
- a vertical line is not actually a function (definitely fails Vertical Line Test!)

Ex



Catalogue of Important Functions: Polynomials

Polynomials: $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0,\ n \in \mathbb{N}$

- **Coefficients** of $P(x)$: $c_0, c_1, c_2, \ldots, c_n \in \mathbb{R}$.
- **Degree** of $P(x)$ is the largest n such that $c_n \neq 0$.
- **Domain** of any polynomial function $P(x)$ is $\mathbb{R}$.
- The **roots** of $P(x)$ are all $x \in \mathbb{R}$ such that $P(x) = 0$.

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degree 0: $P(x) = c_0$ $(y = b)$ **(Constant Functions)**

horizontal line

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degree 1: $P(x) = c_1 x + c_0$ $(y = mx + b)$ **(Linear Functions)**

line of slope c_1

degree 2: $P(x) = c_2x^2 + c_1x + c_0$ (Quadratic Functions)

or $f(x) = ax^2 + bx + c$

- if $a > 0$, parabola opens up. $\cup$

  $\Rightarrow$ range = all real numbers at or above the vertex of the parabola

- if $a < 0$, parabola opens down $\cap$

  $\Rightarrow$ range = all real numbers at or below the vertex of the parabola

- coordinates of vertex: $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

- quadratic can have 0, 1, or 2 (real) roots

  at $\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$ $\leftarrow$ the Quadratic Equation

Exercise 1.3. The height $h(t)$ (in metres) of a ball t seconds after it has been thrown up from an initial height of h_0 m above the ground, with an initial velocity of v_0 m/s, is given by the equation

$$h(t) = -4.9t^2 + v_0 t + h_0$$

[on Earth where gravity $= -9.8 \text{ m/s}^2$]

If the ball is dropped from an initial height of 49 m, sketch the graph of ball's height as a function of time. How many seconds does it take for the ball to reach the ground?

ball dropped $\Rightarrow v_0 = 0$ m/s   from 49m $\Rightarrow h_0 = 49$ m

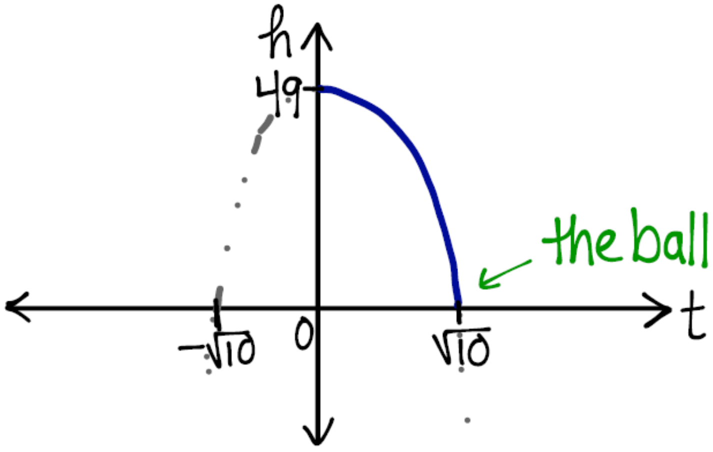
$$h(t) = -4.9t^2 + 49$$

roots: $0 = -4.9t^2 + 49$

$0 = -4.9(t^2 - 10)$

$0 = -4.9(t + \sqrt{10})(t - \sqrt{10})$

$t = -\sqrt{10}$ $t = \sqrt{10}$



the ball reaches the ground when $h(t) = 0$

$\Rightarrow t = \sqrt{10}$ s.

≈ 3.16 s.

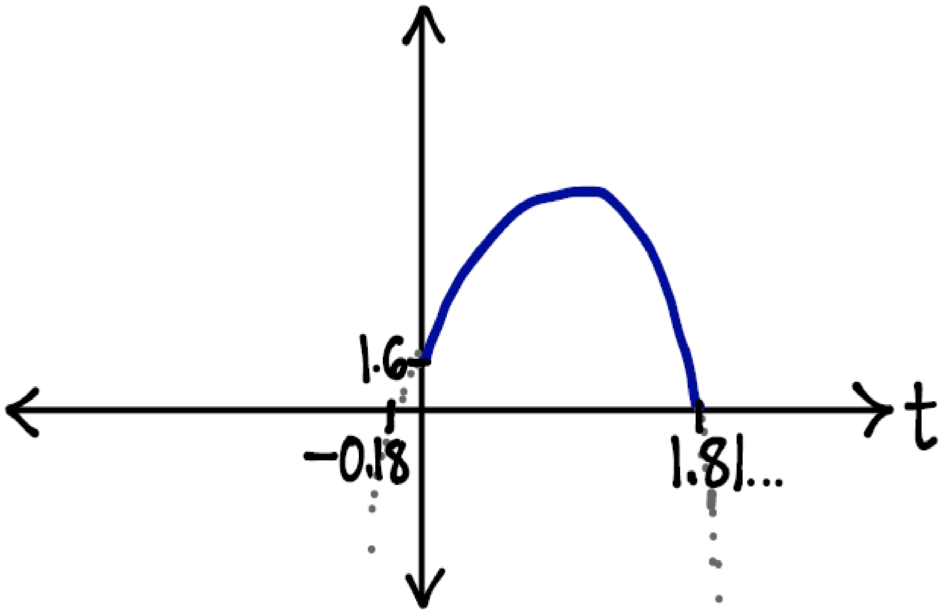
How long until the ball reaches the ground if the ball is thrown upward with an initial velocity of 8 m/s, from an initial height of 1.6 m above the ground?

$V_0 = 8 m/s$ $h_0 = 1.6 m$

$$h(t) = -4.9t^2 + 8t + 1.6$$

roots: $t = \dfrac{-8 \pm \sqrt{8^2 - 4(-4.9)(1.6)}}{2(-4.9)}$

$\Rightarrow t \approx -0.18 s$ or $t \approx 1.81 s$

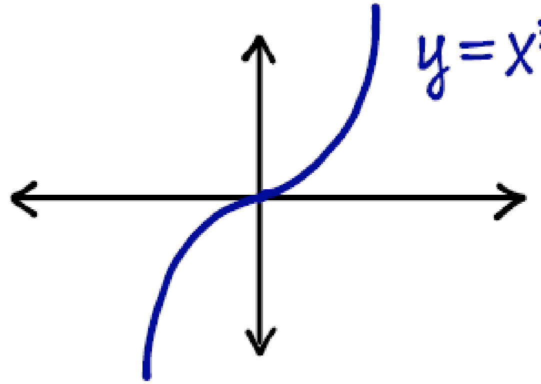


It takes the ball approximately 1.81 seconds to reach the ground.

degree 3: $f(x)=ax^3+bx^2+cx+d$, $a,b,c,d\in\mathbb{R}$, $a\neq 0$ **(Cubic Functions)**

- domain $\mathbb{R}$ • range $\mathbb{R}$
- cubic can have 1, 2, or 3 (real) roots

Ex.



Just like there's a quadratic equation, there is also a Cubic Equation.

The Cubic Equation for finding the roots of $f(x)=ax^3+bx^2+cx+d$:

Let

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$$

$$\Delta_0 = b^2 - 3ac$$

$$\Delta_1 = 2b^3 - 9abc + 27a^2d$$

$$C = \sqrt[3]{\frac{\Delta_1 \pm \sqrt{{\Delta_1}^2 - 4{\Delta_0}^3}}{2}}$$

$$\zeta = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i$$

Then the roots of $f(x)=ax^3+bx^2+cx+d$ are given by

$$x_k = -\frac{1}{3a}\left(b + \zeta^k C + \frac{\Delta_0}{\zeta^k C}\right) \quad \text{for } k=0,1,2$$

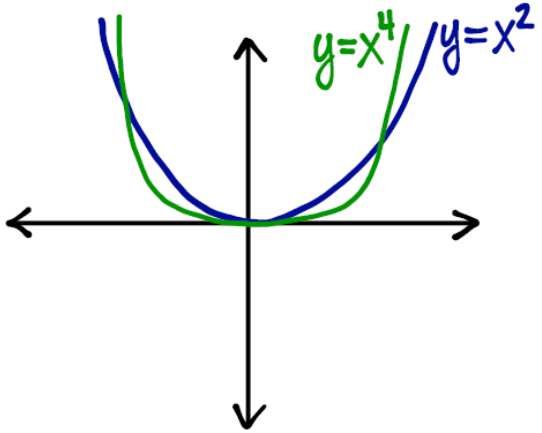
In fact, if $\Delta > 0$, then f has 3 distinct real roots.
If $\Delta = 0$, then f has a multiple root and all its roots are real.
If $\Delta < 0$, then f has 1 real root, and 2 imaginary roots (which are complex conjugates of each other).

Power Functions: $f(x) = x^a$ where a is a constant

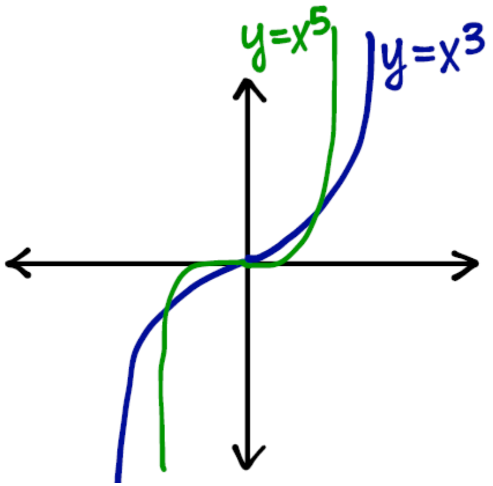
$a = n$ **where n is a positive integer:** $f(x) = x^n$ where n is a positive integer

- domain: $(-\infty, \infty)$

n even



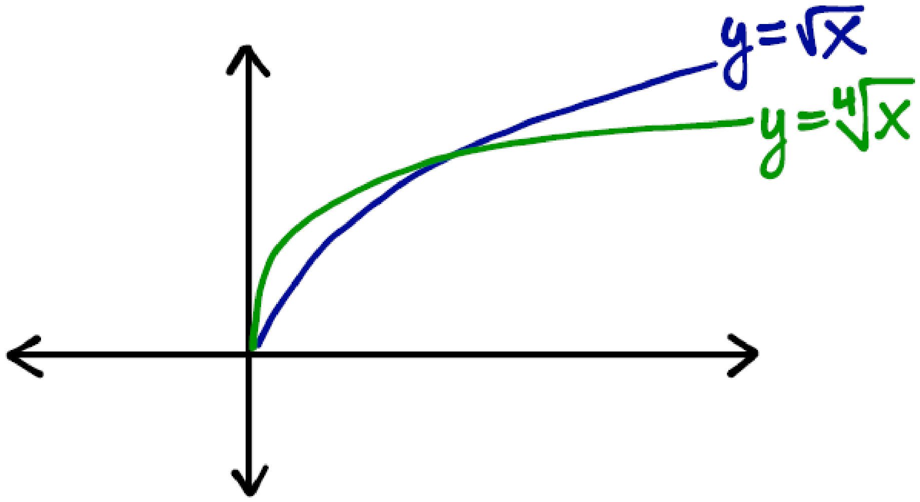
n odd



$a = \frac{1}{n}$ **where n is a positive integer:** $f(x) = x^{1/n} = \sqrt[n]{x}$

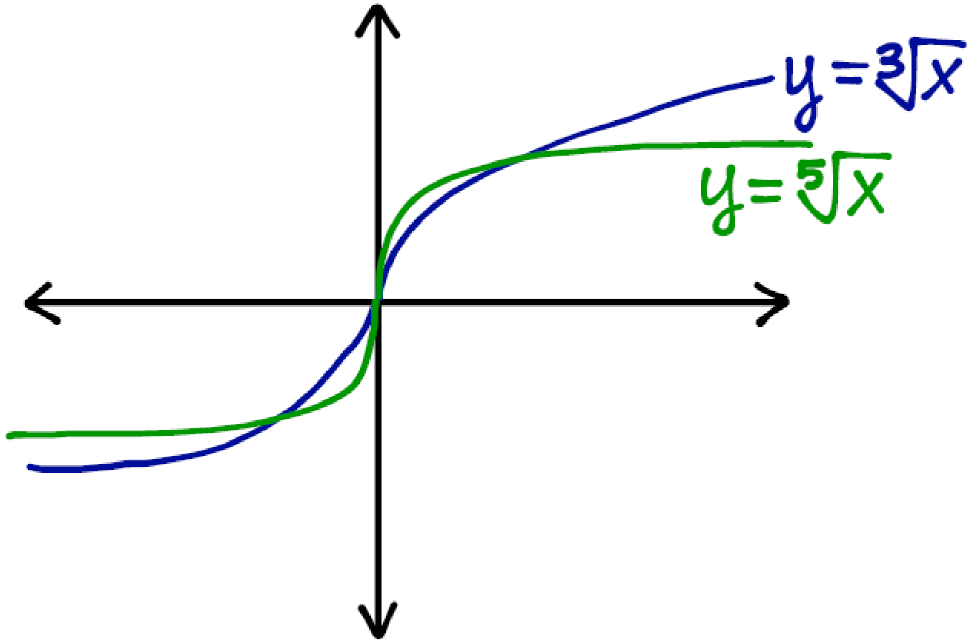
n even

domain: $[0, \infty)$



n odd

domain: $(-\infty, \infty)$

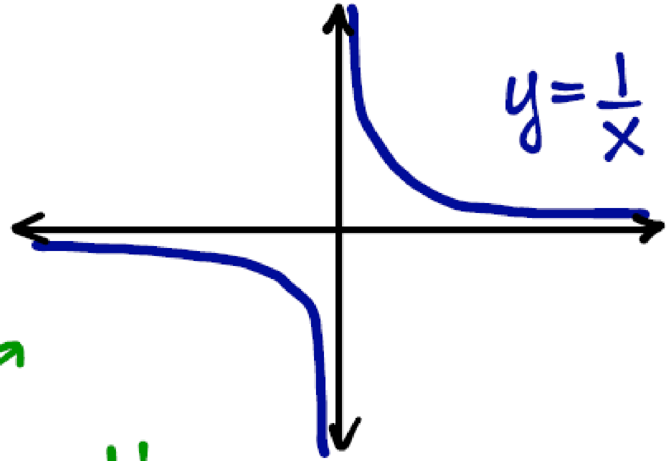


Rational Functions: $f(x) = \dfrac{P(x)}{Q(x)}$ P(x) and Q(x) are polynomials

$Q(x) \neq 0$ (Q(x) is not the constant function $y=0$)

- domain: all real #s except the root(s) (if any) of Q(x)

Ex.



$y = \frac{1}{x}$ ← numerator is polynomial of degree 0
← denominator is polynomial of degree 1

For $y = \frac{1}{x}$, domain is $\{x \in \mathbb{R} : x \neq 0\}$

In interval notation, domain is $(-\infty, 0) \cup (0, \infty)$

note. $y = \frac{1}{x}$ has odd symmetry

Example 1.5. Find the domain of $g(x) = \dfrac{-x^2 - 4x + 5}{x^2 - 1}$. Does the graph of g have any holes or vertical asymptotes?

$$g(x) = \frac{-x^2-4x+5}{x^2-1} = \frac{-x^2-4x+5}{(x+1)(x-1)}$$

denom. $= 0$ when $x = -1$ or $x = 1$

domain of $g = \{x \in \mathbb{R} : x \neq \pm 1\}$

In interval notation,

domain of $g = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

If we factor the numerator of g, we get

$$g(x) = -\frac{(x+5)(x-1)}{(x+1)(x-1)}$$

If x is any real # except 1, then these two factors are nonzero and can be cancelled

$\Rightarrow$ everywhere except at $x=1$, $g(x)$ looks like $y = \dfrac{-(x+5)}{(x+1)}$

__Note.__ $g(x) \neq \dfrac{-(x+5)}{(x+1)}$

$\boxed{\text{compare the domains of } g(x) \text{ and } y = -\dfrac{(x+5)}{(x+1)}}$

$\Rightarrow$ g has a hole at $x=1$ and g has a vertical asymptote at $x=-1$

Algebraic Functions: constructed by combining polynomials using "algebraic" operations

$$+, -, \times, \div, \sqrt[n]{}$$

- can be weird...
- domain depends on roots and denominators...

Ex. $h(x) = x^{2/3}(x-2)^2$

Ex. all rational functions are algebraic

Example 1.6. Find the domain of $g(x) = \sqrt[4]{x^2 - 25}$.

what's inside the root

for an even root function, we need the radicand to be ≥ 0.

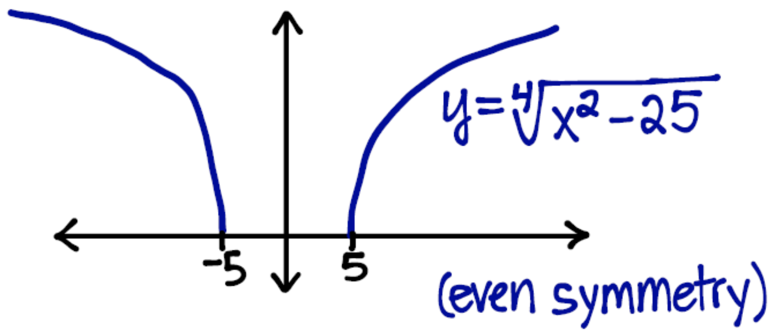
$\Rightarrow$ need $x^2 - 25 \geq 0$

$\Rightarrow \quad x^2 \geq 25$

$\Rightarrow \quad |x| \geq 5$

$\Rightarrow x \in (-\infty, -5] \cup [5, \infty)$

$\therefore$ domain of $g = (-\infty, -5] \cup [5, \infty)$



Exercise 1.7. Find the domain of $f(x) = \dfrac{1}{\sqrt{3 - \frac{2}{x}}} + x^5 - \sqrt{2}$.

we need the radicand $3 - \frac{2}{x} > 0$ and for the fraction $\frac{2}{x}$, we need $x \neq 0$.

to solve the inequality $3 > \frac{2}{x}$ there are 2 cases we need to consider:

Case 1

$x > 0$

so $3 > \frac{2}{x}$

$\Rightarrow 3x > 2$

$\Rightarrow x > \frac{2}{3}$

(and $x > 0$)

Case 2

$x < 0$

so $3 > \frac{2}{x}$

$\Rightarrow 3x < 2$ (multiply both sides of inequality by a negative # $\Rightarrow$ inequality flips)

$\Rightarrow x < \frac{2}{3}$

(and $x < 0$)

$\therefore$ domain of f is $(-\infty, 0) \cup (\frac{2}{3}, \infty)$

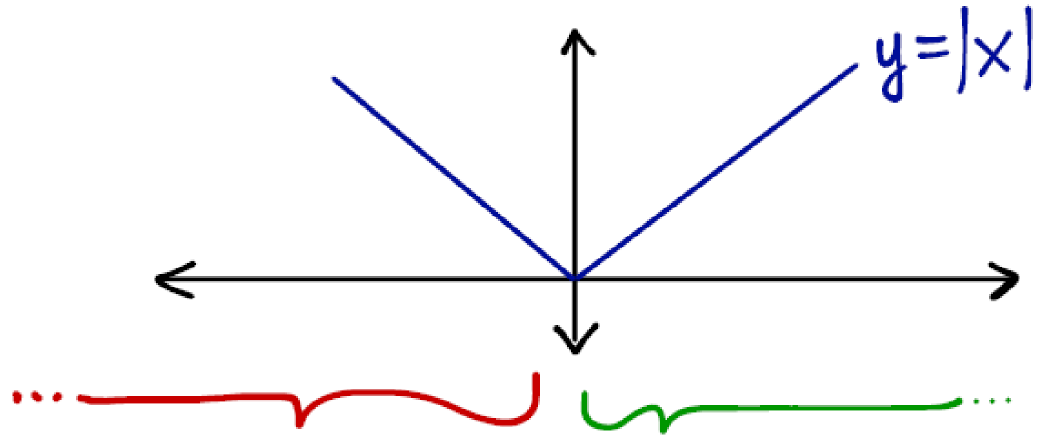
Absolute Value — A Piecewise-defined Function

Absolute Value: $f(x) = |x|$ where $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Absolute Value is a <u>Piecewise</u> defined function

- domain: $(-\infty, \infty)$
- range: $[0, \infty)$

only non-negative numbers are in the range of $|x|$

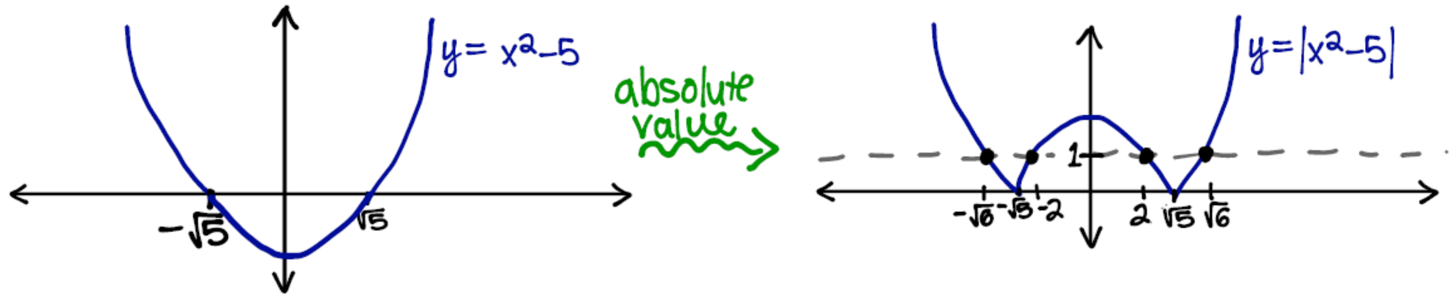


$\boxed{f(x) = |x| \text{ has even symmetry}}$

on the "piece" of the real # line where $x < 0$, the graph of $|x|$ just looks like the line $y = -x$

on the "piece" of the real # line where $x \geq 0$, the graph of $|x|$ just looks like the line $y = x$

Exercise 1.8. Sketch the graph of $g(x) = |x^2 - 5|$. For what values of x is $g(x) = 1$?



$|x^2-5| = 1$

<u>Case 1</u>. $x^2 - 5 \geq 0$

$\therefore |x^2-5| = x^2 - 5$ and $x \in (-\infty, -\sqrt{5}] \cup [\sqrt{5}, \infty)$ (by solving when $x^2 - 5 \geqslant 0$)

So $|x^2-5| = 1 \Rightarrow x^2 - 5 = 1 \Rightarrow x^2 - 6 = 0 \Rightarrow x = -\sqrt{6}$ or $x = \sqrt{6}$

<u>Note</u>. $-\sqrt{6} \in (-\infty, -\sqrt{5}]$ and $\sqrt{6} \in [\sqrt{5}, \infty)$

<u>Case 2</u>. $x^2 - 5 < 0$

$\therefore |x^2-5| = -(x^2-5)$ and $x \in (-\sqrt{5}, \sqrt{5})$ (by solving when $x^2 - 5 < 0$)

So $|x^2-5| = 1 \Rightarrow -(x^2-5) = 1 \Rightarrow x^2 - 4 = 0 \Rightarrow x = -2$ or $x = 2$

<u>Note</u>. $-2 \in (-\sqrt{5}, \sqrt{5})$ and $2 \in (-\sqrt{5}, \sqrt{5})$

$\therefore$ sol$^{\text{ns}}$ to $|x^2-5| = 1$ are $x = -\sqrt{6}$, $x = -2$, $x = 2$, and $x = \sqrt{6}$

STUDY GUIDE & EXERCISES

Stewart, 8th ed.	§1.1 pg. 19 # 3, 7–10, 14, 21, 25, 31–41, 45, 47, 49, 69, 73–77 §1.2 pg. 33 # 1, 3, 11, 17 §1.3 pg. 42 # 1–7, 9, 11, 15, 17, 21, 29ac, 59