# MAT1341 - DGD2 - Solutions
January 24

**1.** Find the general solution to the following system of equations :

$$\begin{aligned} x+y+z &= 0 \\ x-2z &= 0 \\ -x+3y+11z &= 0 \end{aligned}$$

**Solutions :**

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ -1 & 3 & 11 & 0 \end{array}\right] \sim R_2 \to R_2 - R_1 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -3 & 0 \\ -1 & 3 & 11 & 0 \end{array}\right]$$

$$\sim R_3 \to R_3 + R_1 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 4 & 12 & 0 \end{array}\right] \sim R_2 \to -R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 4 & 12 & 0 \end{array}\right]$$

$$\sim R_3 \to R_3 - 4R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim R_1 \to R_1 - R_2 \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The equations are now $x-2z=0$ and $y+3z=0$. $z$ is the free variable, so we isolate the other variables from the equations : $x=2z$ and $y=-3z$. We get the general solution :

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2z \\ -3z \\ z \end{bmatrix} = z \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

where $z \in \mathbb{R}$.

**2.** Consider the following augmented matrix :

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & k & k \end{array}\right]$$

**a)** For what values of $k$ is the system inconsistent ?

**b)** For what values of $k$ does this system have a unique solution ?

**c)** For what values of $k$ does this system have infinitely many solutions ?

**Solutions :**

We first get the system to REF :

$$\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 1 & | & 2 \\ 0 & 2 & k & | & k \end{bmatrix} \sim R_3 \to R_3 - 2R_2 \sim \begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & k-2 & | & k-4 \end{bmatrix}$$

We now see that the system is inconsistent only if $k-2=0$ and $k-4 \neq 0$, which means that the the system is inconsistent when $k=2$. If $k \neq 2$, then

$$\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & k-2 & | & k-4 \end{bmatrix} \sim R_3 \to \frac{1}{k-2} R_3 \sim \begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & 1 & | & \frac{k-4}{k-2} \end{bmatrix}$$

We see that this system has a unique solution (pivots in every column of the coefficient matrix, and no additional pivots in the augmented matrix). In conclusion,

**a)** Inconsistent when $k=2$.

**b)** A unique solution when $k \neq 2$.

**c)** No values of $k$ gives the system infinitely many solutions.

**3.** Consider a linear system with 600 equations and 1000 variables. Determine if the following statements are true or false :

**a)** Such a system is always consistent.

**b)** Such a system always has a unique solution.

**c)** Such a system can be inconsistent.

**d)** If such a system is consistent, the it always has a unique solution.

**e)** The general solution to such a system has exactly 400 non-leading variables.

**f)** The general solution to such a system can have more than 400 non-leading variables.

**g)** If the system is homogeneous, then it has infinitely many solutions.

**Solutions :**

**a)** False. The system could have the equation $0 = 1$ as one of its 600 equations.

**b)** False. See a), or see that since there are more equations than variables, there cannot be a pivot in every column.

**c)** True, see a).

**d)** False, if it is consistent, then there can only be at most 600 pivots, since there is only at most one pivot per row. Therefore, there will be free variables.

**e)** False. This is the case only if every row has a pivot.

**f)** True. The system could have only 1 pivot, leading to 999 free variables.

**g)** True. Any homogeneous system is consistent. In this case, there cannot be a pivot in every column (not enough rows), so it will have infinitely many solutions.

**4.** Find the reduced row echelon form of the following matrix :

$$\begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 2 \\ 4 & -2 & 1 \end{bmatrix}$$

**Solutions :**

$$\begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 2 \\ 4 & -2 & 1 \end{bmatrix} \sim R_2 \leftrightarrow R_1 \sim \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 3 \\ 4 & -2 & 1 \end{bmatrix} \sim R_1 \to \frac{1}{2} R_1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & -2 & 1 \end{bmatrix} \sim R_3 \to R_3 - 4R_1 \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\sim R_2 \to \frac{1}{2} R_2 \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3/2 \\ 0 & -2 & -3 \end{bmatrix} \sim R_3 \to R_3 + 2R_2 \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$