MAT1341 - DGD3 - Solutions
January 31

1. Consider the following system of linear equations with the variables x and y :

$$\begin{aligned} x+2y &= 3 \\ 3x+2ay &= b \end{aligned}$$

For what values of a and b does this system have

a) No solutions ?

b) A unique solution ?

c) Infinitely many solutions ?

Solutions :

The augmented matrix for this system is $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 2a & b \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 2a-6 & b-9 \end{array}\right]$.

a) This system has no solution if the last row has a pivot in the last column. This is achieved by setting $2a-6=0$ and $b-9\neq 0$. So this system has no solutions if $a=3$ and $b\neq 9$.

b) This system has a unique solution if there is a pivot in the second column. This is achieved by setting $2a-6\neq 0$ and b to be any real number. So this system has no solutions if $a\neq 3$ and b is any real number.

c) This system has infinitely many solutions if it is consistent and if the second column has no pivot. This is achieved by setting $2a-6=0$ and $b-9=0$. So this system has infinitely many solutions if $a=3$ and $b=9$.

2. Let $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$. Find AB and BA, if they exist.

Solutions :

$$AB=\begin{bmatrix} 8 & 5 \\ 20 & 11 \end{bmatrix}$$

$$BA=\begin{bmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \\ 5 & 7 & 9 \end{bmatrix}$$

3. Consider the following linear system with the three variables x,y and z :

$$\begin{aligned} x+y+2z &= 1 \\ 2x+y+az &= 2 \\ 3x+4y+az &= c \end{aligned}$$

a) For what values of a and c is this system inconsistent ?

b) For what values of a and c does this system have a unique solution ?

c) For what values of a and c does this system have infinitely many solutions ?

d) Using your answer in c), find the general solution when this system has infinitely many solutions.

Solutions :

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 1 & a & 2 \\ 3 & 4 & a & c \end{array}\right] \sim_{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & a-4 & 0 \\ 3 & 4 & a & c \end{array}\right] \sim_{R_3 \to R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & a-4 & 0 \\ 0 & 1 & a-6 & c-3 \end{array}\right]$$

$$\sim_{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 4-a & 0 \\ 0 & 1 & a-6 & c-3 \end{array}\right] \sim_{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 4-a & 0 \\ 0 & 0 & 2a-10 & c-3 \end{array}\right]$$

a) The system is inconsistent if $2a-10=0$ and $c-3 \neq 0$, so if $a=5$ and $c \neq 3$.

b) The system has a unique solution if $2a-10 \neq 0$, so if $a \neq 5$, and c is any real number.

c) The system has infinitely many solutions if $2a-10=0$ and if $c-3=0$, so if $a=5$ and $c=3$.

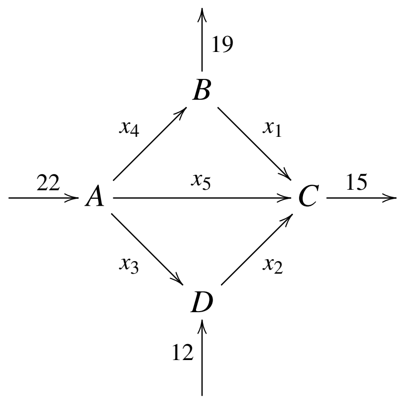
d) If $a=5$ and $c=3$, the matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim_{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is therefore $x = 1-3z$, $y=z$ and z is free. In vector form :

$$\begin{bmatrix} 1-3z \\ z \\ z \end{bmatrix}, \qquad z \in \mathbb{R}$$

4. Consider the network of streets with intersections A, B, C and D below. The arrows indicate the direction of traffic flow along the one-way streets, and the numbers refer to the exact number of cars observed to enter or leave A, B, C and D during one minute. Each x_i denotes the unknown number of cars which passes along the indicated streets during the same period.



a) Write the system of equations representing this network, and give all constraints on the variables x_i, $i = 1, \ldots, 5$.

b) Find the general solution to this system.

c) If the road AC is closed, what is the minimum and maximum flow along road DC ?

Solutions :

a) We use flow in = flow out :

$$\begin{aligned} 22 &= x_3 + x_4 + x_5 \\ x_4 &= 19 + x_1 \\ x_1 + x_2 + x_5 &= 15 \\ x_3 + 12 &= x_2 \end{aligned}$$

Where each x_i is a non negative integer.

b) This system becomes the following matrix :

$$\left[\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 1 & 22 \\ 1 & 0 & 0 & -1 & 0 & -19 \\ 1 & 1 & 0 & 0 & 1 & 15 \\ 0 & 1 & -1 & 0 & 0 & 12 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & -19 \\ 0 & 1 & 0 & 1 & 1 & 34 \\ 0 & 0 & 1 & 1 & 1 & 22 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The variables x_4 and x_5 are free, so let $x_4 = t$ and $x_5 = s$. The general solution is $x_1 = t - 19, x_2 = 34 - t - s, x_3 = 22 - t - s, x_4 = t$ and $x_5 = s$. In vector form :

$$\begin{bmatrix} t-19 \\ 34-t-s \\ 22-t-s \\ t \\ s \end{bmatrix}, \qquad s,t \in \mathbb{R}$$

c) Since AC is closed, we have $x_5 = s = 0$. We would like to know the maximum and minimum value for x_2. We have $x_1 = t - 19 \geq 0$, so $t \geq 19$. We have $x_3 = 22 - t \geq 0$, so $t \leq 22$. So $19 \leq t \leq 22$, and so x_2 is minimal when t is maximal : the minimum value for x_2 is $34 - 22 = 12$. x_2 is maximal when t is minimal : the maximum value for x_2 is $34 - 19 = 15$.