Solution to the Final Examination

MAT1322D, Fall 2017

Part I. Multiple-choice Questions ($2 \times 12 = 24$ marks)

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1. The area of the region above the graph of $y = x^2$ and under the graph of $y = 8 - x^2$ in the first quadrant is

(A) $\frac{27}{3}$; (B) $\frac{29}{3}$; (C) $\frac{37}{3}$; (D) $\frac{41}{3}$; (E) $\frac{35}{3}$; (F) $\frac{32}{3}$.

Solution. (F) Let $x^2 = 8 - x^2$. $2x^2 = 8$, $x = 2$. The area is

$$\int_0^2 (8 - x^2 - x^2)dx = 2\int_0^2 (4 - x^2)dx = 2\left[4x - \frac{x^3}{3}\right]_{x=0}^2 = \frac{32}{3}.$$

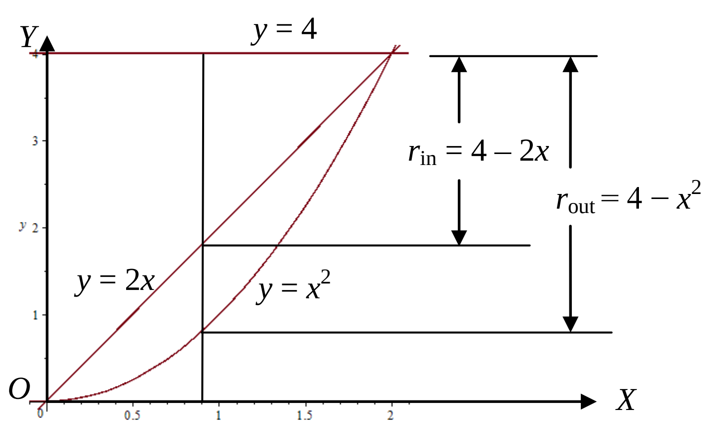
2. Let R be the region above the parabola $y = x^2$ and under the line $y = 2x$. Solid B is obtained by revolving R about the line $y = 4$. Then the volume of B is calculated by the integral

(A) $\pi\int_0^2 ((4 - x^2)^2 - (4 - 2x)^2)dx$; (B) $\pi\int_0^4 ((4 - x^2)^2 - (4 - 2x)^2)dx$;

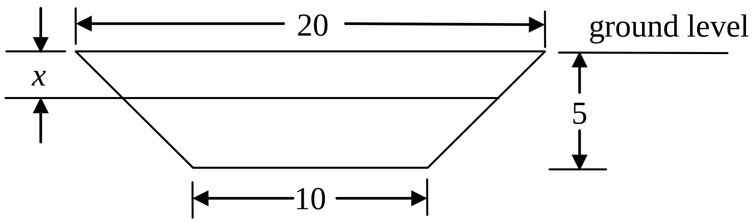
(C) $\pi\int_0^2 ((4 + x^2)^2 - (4 + 2x)^2)dx$; (D) $\pi\int_0^4 ((4 + x^2)^2 - (4 + 2x)^2)dx$;

(E) $\pi\int_0^2 ((4 - 2x)^2 - (4 - x^2)^2)dx$; (F) $\pi\int_0^2 ((4 + x^2)^2 - (4 - 2x)^2)dx$.

Answer. (A)



3. Suppose a pool has the shape of an inverted truncated pyramid. The top of the pool at the ground level is a square with side length 20 meters, and the bottom of the pool is a square with side length 10 meters. The depth of the pool is 5 meters. The pool is filled with water with density ρ kg / m^3. Let x be the distance between a horizontal layer of water and the top of the pool.



Denote the acceleration of gravity be g m / sec^2. Then the work, in Joules, needed to pump the water in the pool to a point 2 meters above the ground is calculated by the integral

(A) $\rho g\int_0^5 (20-2x)^2(7-x)dx$; (B) $\rho g\int_0^7 (20-2x)^2(x+2)dx$

(C) $\pi\rho g\int_0^7 (20-2x)^2(7-x)dx$; (D) $\rho g\int_0^5 (20-2x)^2(x+2)dx$

(E) $\pi\rho g\int_0^5 (20-2x)^2(7-x)x\,dx$; (F) $\pi\rho g\int_0^5 (20-2x)^2(x+2)dx$

Solution. (D) A horizontal layer of water in the pool is a square with side-length $L(x) = 10 + 2\,(5 - x) = 20 - 2x$.

The volume of this layer with thickness dx is $V(x) = (L(x))^2 = (20 - 2x)^2dx$.

The weight of this layer of water is $w(x) = \rho gV(x) = \rho g(20 - 2x)^2dx$.

The work needed to pump this layer of water to a point 2 meters above the ground is

$W(x) = w(x)(x + 2) = \rho g(20 - 2x)^2(x + 2)dx$.

The total work is

$$W = \rho g\int_0^5 (20-2x)^2(x+2)dx .$$

4. Recall that the length of the arc $y = f(x)$, $a \le x \le b$, is calculated by the formula $L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$. Then the length of the arc $y = \ln x - \dfrac{x^2}{8}$, $1 \le x \le e$, is

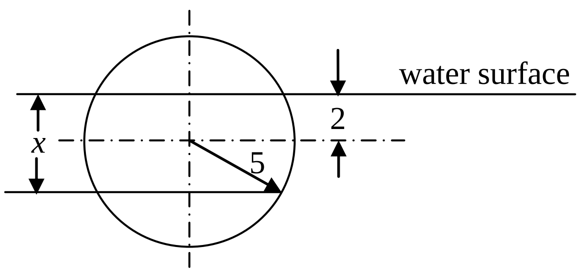
(A) $\frac{1}{8}(e^2+7)$; (B) $\frac{1}{8}(e^2+5)$; (C) $\frac{1}{8}(e^2+1)$;

(D) $\frac{1}{4}(e^2+7)$; (E) $\frac{1}{4}(e^2+5)$; (F) $\frac{1}{4}(e^2+1)$.

Solution. (A) $y' = \frac{1}{x} - \frac{x}{4}$, and $(y')^2 = \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16}$. $1 + (y')^2 = \frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16} = \left(\frac{1}{x} + \frac{x}{4}\right)^2$.

$$L = \int_1^e \left(\frac{1}{x} + \frac{x}{4}\right)dx = \left[\ln x + \frac{x^2}{8}\right]_{x=1}^{e} = 1 + \frac{e^2}{8} - \frac{1}{8} = \frac{1}{8}(e^2+7).$$

5. Suppose a surface of the shape of a disk of radius 5 meters is vertically partially submerged into water, with density ρ kg/m^3, so that the center is 2 meters under the water.



Let x be the depth of a horizontal stripe of the surface. Let g be the acceleration of gravity. Then the force, in Newtons, acting on the disk is calculated by the integral

(A) $2\rho g\int_{-2}^{5} x\sqrt{5^2-(x-5)^2}\,dx$; (B) $2\rho g\int_{0}^{7} (x+2)\sqrt{5^2-(x-5)^2}\,dx$;

(C) $2\rho g\int_{3}^{10} (x+2)\sqrt{5^2-x^2}\,dx$; (E) $2\rho g\int_{-2}^{5} x\sqrt{5^2-(x-2)^2}\,dx$;

(D) $2\rho g\int_{3}^{10} x\sqrt{5^2-x^2}\,dx$; (F) $2\rho g\int_{0}^{7} x\sqrt{5^2-(x-2)^2}\,dx$.

Solution. (F) The area of a horizontal stripe with height dx on the surface at depth x is $A(x) = 2\sqrt{5^2-(x-2)^2}\,dx$. The pressure on this stripe is $P(x) = \rho g x$. The force acting on this this stripe is $F(x) = A(x)P(x) = 2\rho g x\sqrt{5^2-(x-2)^2}\,dx$. The total force is $F = 2\rho g\int_0^7 x\sqrt{5^2-(x-2)^2}\,dx$.

6. The centroid of the region under the parabola $y = x - x^2$ and above the x-axis is

(A) $\left(\frac{1}{2}, \frac{1}{12}\right)$; (B) $\left(\frac{1}{5}, \frac{1}{2}\right)$; (C) $\left(\frac{1}{2}, \frac{1}{15}\right)$;

(D) $\left(\frac{1}{12}, \frac{1}{2}\right)$; (E) $\left(\frac{1}{2}, \frac{1}{10}\right)$; (F) $\left(\frac{1}{10}, \frac{1}{5}\right)$.

Solution. (E) The mass is $m = \int_0^1 (x - x^2)dx = \frac{1}{6}$. The moments are

$M_x = \frac{1}{2}\int_0^1 (x - x^2)^2 dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_{x=0}^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{1}{60}$.

$M_y = \int_0^1 x(x - x^2)dx = \frac{1}{12}$.

Then $\bar{x} = \frac{1/12}{1/6} = \frac{1}{2}$, $\bar{y} = \frac{1/60}{1/6} = \frac{1}{10}$. The centroid is $\left(\frac{1}{2}, \frac{1}{10}\right)$.

7. Consider improper integral $\int_0^1 \frac{\sqrt{x}+1}{x^2+x}dx$. Which one of the following argument is true?

(A) Since $\frac{\sqrt{x}+1}{x^2+x} < \frac{2}{x}$ in (0, 1) and $\int_0^1 \frac{2}{x}dx = 2\int_0^1 \frac{1}{x}dx$ converges, $\int_0^1 \frac{\sqrt{x}+1}{x^2+x}dx$ converges.

(B) Since $\frac{\sqrt{x}+1}{x^2+x} < \frac{2}{x}$ in (0, 1) and $\int_0^1 \frac{2}{x}dx = 2\int_0^1 \frac{1}{x}dx$ diverges, $\int_0^1 \frac{\sqrt{x}+1}{x^2+x}dx$ diverges.

(C) Since $\frac{\sqrt{x}+1}{x^2+x} < \frac{2\sqrt{x}}{x}$ in (0, 1) and $\int_0^1 \frac{2\sqrt{x}}{x}dx = 2\int_0^1 \frac{1}{x^{1/2}}dx$ converges, $\int_0^1 \frac{\sqrt{x}+1}{x^2+x}dx$ converges.

(D) Since $\frac{\sqrt{x}+1}{x^2+x} > \frac{\sqrt{x}}{2x^2}$ in (0, 1) and $\int_0^1 \frac{\sqrt{x}}{2x^2}dx = \frac{1}{2}\int_0^1 \frac{1}{x^{3/2}}dx$ diverges, $\int_0^1 \frac{\sqrt{x}+1}{x^2+x}dx$ diverges.

(E) Since $\frac{\sqrt{x}+1}{x^2+x} > \frac{1}{2x}$ in (0, 1) and $\int_0^1 \frac{1}{2x}dx = \frac{1}{2}\int_0^1 \frac{1}{x}dx$ diverges, $\int_0^1 \frac{\sqrt{x}+1}{x^2+x}dx$ diverges.

(F) Since $\frac{\sqrt{x}+1}{x^2+x} > \frac{1}{2x}$ in (0, 1) and $\int_0^1 \frac{1}{2x}dx = \frac{1}{2}\int_0^1 \frac{1}{x}dx$ converges, $\int_0^1 \frac{\sqrt{x}+1}{x^2+x}dx$ converges.

Answer. (E).

8. Suppose Euler's method with step size $h = 0.05$ is used to find an approximation of $y(0.1)$, where $y(t)$ is the solution to the initial-value problem $y' = (2t - 1)(y + 1)$, $y(0) = 1$. Which one of the following is closest to the answer? $y(0.1) \approx$

(A) 0.905; (B) 0.845; (C) 0.815; (D) 0.742; (E) 0.707; (F) 0.685.

Solution. (C)

i	t_i	y_i
0	0	1
1	0.05	$1 + 0.05 \times (2 \times 0 - 1) \times (1 + 1) = 0.900$
2	0.10	$0.900 + 0.05 \times (2 \times 0.05 - 1) \times (0.9 + 1) = 0.815$

9. The sum of the series $\sum_{n=0}^{\infty} \frac{2^{n+1} + (-1)^n 5^n}{3^{2n}}$ is $S =$

(A) $\frac{43}{14}$; (B) $\frac{45}{14}$; (C) $\frac{29}{9}$; (D) $\frac{25}{7}$; (E) $\frac{13}{9}$; (F) $\frac{23}{7}$.

Solution. (B) $\sum_{n=0}^{\infty} \frac{2^{n+1} + (-1)^n 5^n}{3^{2n}} = \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{2n}} + \sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{3^{2n}}$. The first series is a geometric series with first term $a_1 = 2$ and common ratio $r_1 = \frac{2}{9}$. The second series is also a geometric series with first term $a_2 = 1$ and common ratio $r_2 = -\frac{5}{9}$. Hence,

$$S = \frac{2}{1 - \frac{2}{9}} + \frac{1}{1 + \frac{5}{9}} = \frac{18}{7} + \frac{9}{14} = \frac{45}{14}$$

10. The interval of convergence of the series $\sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n$ is

(A) $-1.5 \le x \le 1.5$; (B) $-1.5 < x < 1.5$; (C) $-1.5 < x \le 1.5$;
(D) $-1.5 \le x < 1.5$; (E) $-\infty < x < \infty$; (F) only at $x = 0$.

Solution. (B) $\lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1}}{3^{n+1}} \frac{3^n}{2^n x^n} \right| = \lim_{n\to\infty} \left| \frac{2x}{3} \right| = \frac{2}{3}|x|$. When $|x| < \frac{3}{2}$, this series is absolutely convergent. When $|x| > \frac{3}{2}$, this series is divergent. When $x = -\frac{3}{2}$, this series becomes $\sum_{n=0}^{\infty} \frac{2^n}{3^n} \left(-\frac{3}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n$; when $x = \frac{3}{2}$, this series becomes $\sum_{n=0}^{\infty} \frac{2^n}{3^n} \left(\frac{3}{2}\right)^n = \sum_{n=0}^{\infty} 1$. In both cases, the series diverges. Hence, the interval of convergence is $\left(-\frac{3}{2}, \frac{3}{2}\right) = (-1.5, 1.5)$.

11. If $f(x, y) = x^2y^2 + 3xy + y^3$, then the gradient vector of $f(x, y)$ at the point $x = 2$, $y = -1$ is

(A) $(1, -1)$; (B) $(-1, 1)$; (C) $(-1, -1)$; (D) $(1, 1)$;
(E) $(3, 1)$; (F) $(1, 3)$.

Solution. (D) $f_x = 2xy^2 + 3y$, $f_y = 2x^2y + 3x + 3y^2$. When $x = 2$ and $y = -1$, grad $f(2, -1) = (1, 1)$.

12. The directional derivative of the function $z = \ln(2x^2 - y^2)$ at the point $x = 3$ and $y = 4$ in the direction of the vector $\mathbf{u} = (3, 4)$ is

(A) 1; (B) $-\frac{2}{5}$; (C) $-\frac{1}{5}$; (D) $\frac{2}{5}$; (E) -2; (F) 2.

Solution. (D) The unit vector in the direction of $\mathbf{u}$ is $\mathbf{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$. $z_x = \frac{4x}{2x^2 - y^2}$, $z_y = -\frac{2y}{2x^2 - y^2}$. When $x = 3$ and $y = 4$, $z_x(3, 4) = 6$, $z_y(3, 4) = -4$. $D_{\mathbf{u}}(z) = 6\cdot\frac{3}{5} + (-4)\cdot\frac{4}{5} = \frac{2}{5}$.

Part II. Long Answer Questions (26 marks)

1. (5 marks) **Use the definition of improper integrals** to determine whether improper integral $\int_0^\infty \frac{x}{(x^2+1)^2}\,dx$ is convergent or divergent. If it is convergent, find its value.

Solution. $\int_0^\infty \frac{x}{(x^2+1)^2}\,dx = \lim_{b\to 0}\int_0^b \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\lim_{b\to 0}\int_1^{b^2+1}\frac{1}{u^2}\,du = \frac{1}{2}\lim_{b\to 0}\left[\frac{1}{1} - \frac{1}{b^2+1}\right] = \frac{1}{2}$. This improper integral is convergent, and its value is $\frac{1}{2}$.

2. (5 marks) Find function $y(t)$, where $y(t)$ is the solution to the initial-value problem $y' = y\sin t$, $y(0) = -1$.

Solution. $\int \frac{1}{y}\,dy = \int \sin t\,dt$. $\ln|y| = -\cos t + C$, $|y| = K_1e^{-\cos t}$, where $K_1 = e^C > 0$. Then $y = Ke^{-\cos t}$, where $K = \pm K_1 \neq 0$. When $t = 0$, $\cos t = 1$. $y(0) = Ke^{-1} = -1$. Then $K = -e$. Hence, $y(t) = -e\cdot e^{-\cos t} = -e^{1-\cos t}$.

3. (6 marks) Use an appropriate test method to determine whether each of the following series is convergent or divergent.

(a) $\sum_{n=1}^{\infty}\frac{2n-\sqrt{n}}{n^2+1}$; (b) $\sum_{n=0}^{\infty}(-1)^n\sqrt{\frac{1}{n^2+1}}$; (c) $\sum_{n=2}^{\infty}\left(\frac{n+1}{3n}\right)^n$.

Solution. (a) Since this series is positive, we can use the limit comparison test. Let $a_n = \dfrac{2n-\sqrt{n}}{n^2+1}$, and let $b_n = \dfrac{1}{n}$. Then $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\dfrac{2n^2-n\sqrt{n}}{n^2+1} = \lim\limits_{n\to\infty}\dfrac{2-1/\sqrt{n}}{1+1/n^2} = 2$. Since series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ diverges, series $\sum\limits_{n=1}^{\infty}\dfrac{2n-\sqrt{n}}{n^2+1}$ diverges.

This question can also be solved by comparison test. $\dfrac{2n-\sqrt{n}}{n^2+1} > \dfrac{n}{2n^2} = \dfrac{1}{2n}$. Since series $\sum\limits_{n=1}^{\infty}\dfrac{1}{2n} = \dfrac{1}{2}\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ diverges, series $\sum\limits_{n=1}^{\infty}\dfrac{2n-\sqrt{n}}{n^2+1}$ diverges.

(b) Since function $f(n) = \sqrt{\dfrac{1}{n^2+1}}$ is decreasing, and $\lim\limits_{n\to\infty}\sqrt{\dfrac{1}{n^2+1}} = 0$, By the alternating series test, this series converges.

(c) Use the root test. Let $a_n = \left(\dfrac{n+1}{3n}\right)^n$. $\lim\limits_{n\to\infty}\sqrt[n]{a_n} = \lim\limits_{n\to\infty}\dfrac{n+1}{3n} = \dfrac{1}{3}$. This series is convergent.

This question may also be solved by the ratio test.

Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{n+2}{3(n+1)}\right)^{n+1}}{\left(\frac{n+1}{3n}\right)^{n}}\right| = \lim_{n\to\infty}\left|\left(\frac{n+2}{3(n+1)}\right)^{n+1}\left(\frac{3n}{n+1}\right)^{n}\right| = \frac{1}{3}\lim_{n\to\infty}\left|\left(1+\frac{1}{n+1}\right)^{n+1}\right|\lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^{-n}\right| = \frac{1}{3}$$

< 1.

This series is (absolutely) convergent.

4. (6 marks) The Maclaurin series of the function $y = \sin x$ is

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots .$$

(a) (4 marks) Find the first three non-zero terms of the Maclaurin series of the function $F(x) = \int_0^x \sin(2t^2)\,dt$.

(b) (2 marks) Find The fifth and the seventh derivative of function $F(x)$ at $x = 0$, i.e., $F^{(5)}(0)$ and $F^{(7)}(0)$.

Solution. (a) $\sin(2t^2) = \sum_{n=0}^{\infty}(-1)^n \frac{(2t^2)^{2n+1}}{(2n+1)!} = 2t^2 - \frac{2^3t^6}{3!} + \frac{2^5t^{10}}{5!} - \ldots$.

$$F(x) = \int_0^x \sin(2t^2)dt = \int_0^x \sum_{n=0}^{\infty}(-1)^n \frac{(2t^2)^{2n+1}}{(2n+1)!}dt = \int_0^x \left(2t^2 - \frac{2^3t^6}{3!} + \frac{2^5t^{10}}{5!} - \ldots\right)dt$$
$$= \frac{2}{3}x^3 - \frac{8}{42}x^7 + \frac{32}{1320}x^{11} - \ldots = \frac{2}{3}x^3 - \frac{4}{21}x^7 + \frac{4}{165}x^{11} - \ldots .$$

(b) $F^{(5)}(0) = 0$, $F^{(7)}(0) = 7!\left(-\frac{4}{21}\right) = -960$.

5. (4 marks) Find the equation of the tangent plane of the graph of the equation $x^2z + xy - yz^3 = -1$ at the point $(2, 1, -1)$.

Solution. Let $F(x, y, z) = x^2z + xy - yz^3 + 1$.

Then $F_x = 2xz + y$, $F_y = x - z^3$, and $F_z = x^2 - 3yz^2$. Hence, $F_x(2, 1, -1) = -3$, $F_y(2, 1, -1) = 3$, and $F_z(2, -1, 3) = 1$.

The equation of the tangent plane at the point $(2, 1, -1)$ is $-3(x - 2) + 3(y - 1) + (z + 1) = 0$, or $3x - 3y - z = 4$.