ENGR 213/T: APPLIED ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

Lecture 8

2.7 Linear Mathematical Models

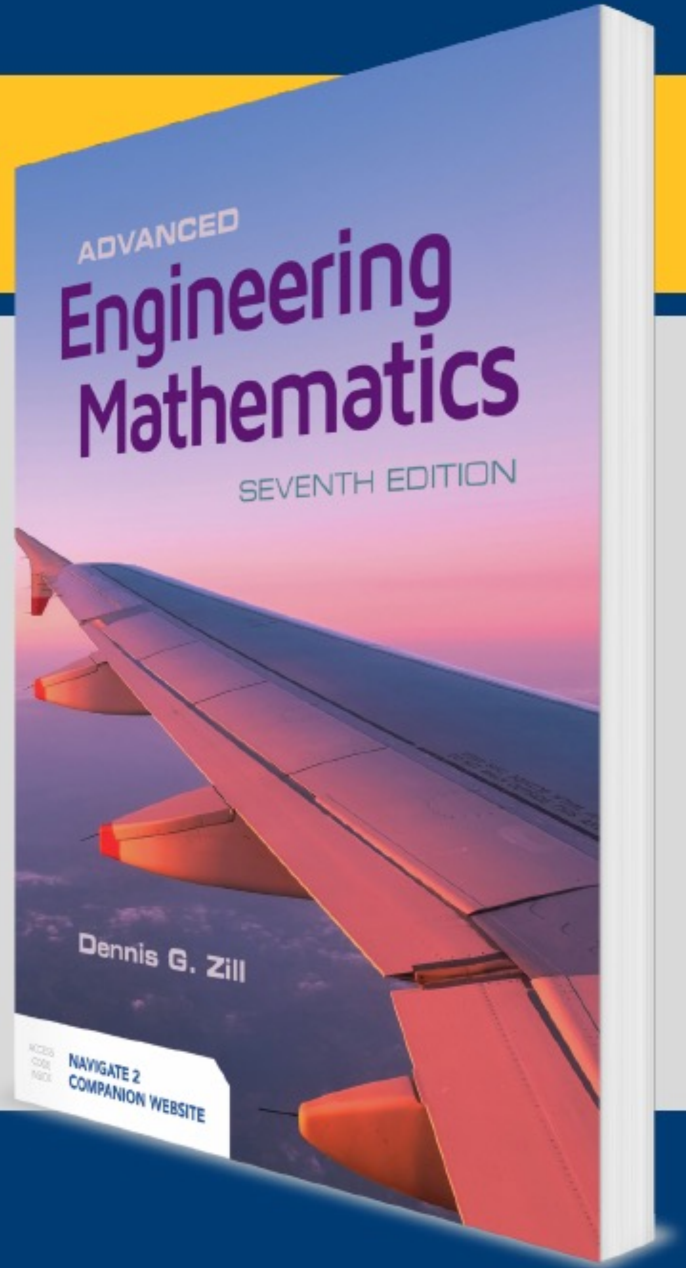
Lecture: TUE THU 10:15 am – 11:30 am (FG-C080 SGW)

Tutorials TA: MON 1:15 pm – 2:55 pm by Niklas (H-621 SGW)
Tutorials TB: MON 1:15 pm – 2:55 pm by Mohsen (FG-B060 SGW)

Instructor: Sam Eskandarian, Ph.D. PEng (Ontario)

CHAPTER 2

First-Order Differential Equations



Outline

1.3 Differential Equations as Mathematical Models

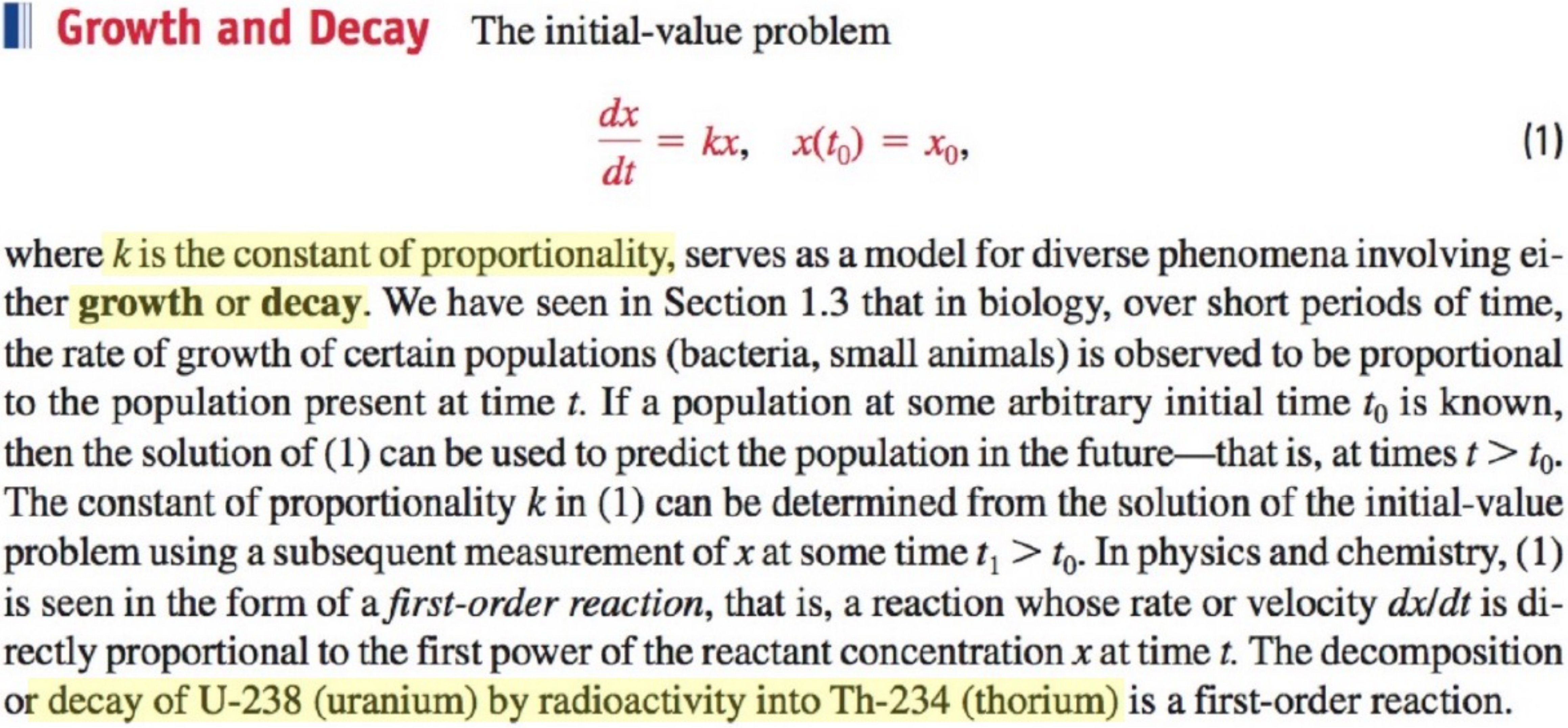
2.7 Linear Models

2.7 Linear Models

2.7 Linear Models

INTRODUCTION In this section we solve some of the linear first-order models that were introduced in Section 1.3.

Linear Models (cont'd.)



Growth and Decay The initial-value problem

$$\frac{dx}{dt} = kx, \quad x(t_0) = x_0, \tag{1}$$

where k is the constant of proportionality, serves as a model for diverse phenomena involving either **growth** or **decay**. We have seen in Section 1.3 that in biology, over short periods of time, the rate of growth of certain populations (bacteria, small animals) is observed to be proportional to the population present at time t. If a population at some arbitrary initial time t_0 is known, then the solution of (1) can be used to predict the population in the future—that is, at times $t > t_0$. The constant of proportionality k in (1) can be determined from the solution of the initial-value problem using a subsequent measurement of x at some time $t_1 > t_0$. In physics and chemistry, (1) is seen in the form of a *first-order reaction*, that is, a reaction whose rate or velocity dx/dt is directly proportional to the first power of the reactant concentration x at time t. The decomposition or decay of U-238 (uranium) by radioactivity into Th-234 (thorium) is a first-order reaction.

Linear Models (cont'd.)

Example

EXAMPLE 1 Bacterial Growth

A culture initially has P_0 number of bacteria. At $t = 1$ h the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

P(0) = P.
P(1)= 3/2 P.

dP/dt = kP ==> DP/dt -kP = 0

U = e^(-kt)
P(t) = Ce^kt --> from P. = P(0), P(t) = P. e^kt --> from this k = 0.405

for P. to triple ==> t = 2.71

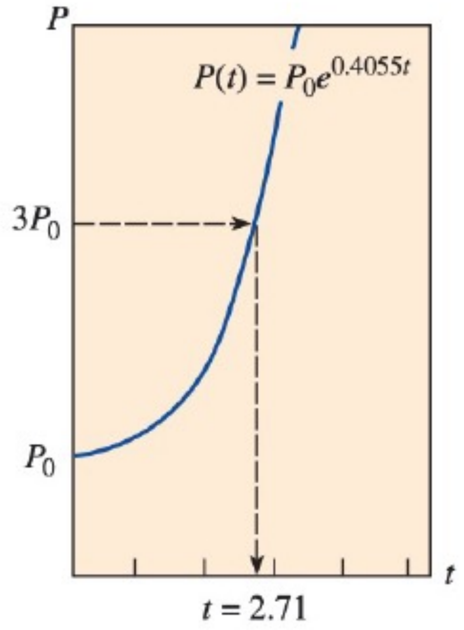


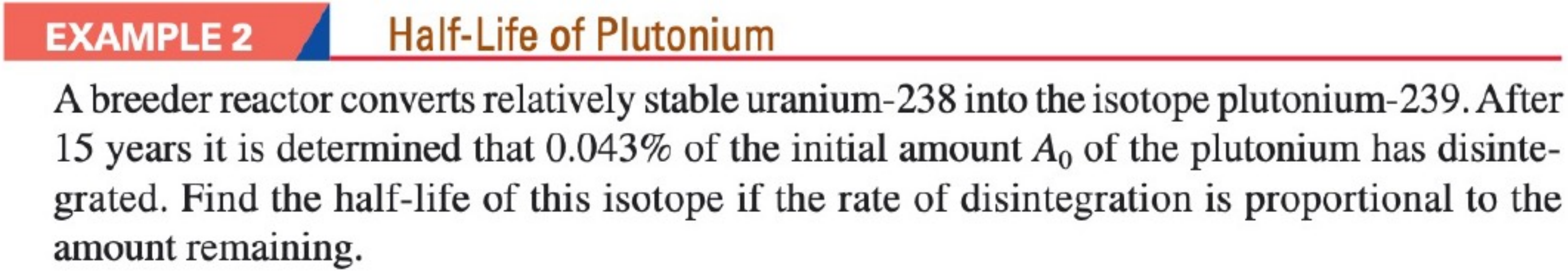
FIGURE 2.7.1 Time in which initial population triples in Example 1

Linear Models (cont'd.)

Half-Life In physics the **half-life** is a measure of the stability of a radioactive substance. The half-life is simply the time it takes for one-half of the atoms in an initial amount A_0 to disintegrate, or transmute, into the atoms of another element. The longer the half-life of a substance, the more stable it is. For example, the half-life of highly radioactive radium, Ra-226, is about 1700 years. In 1700 years one-half of a given quantity of Ra-226 is transmuted into radon, Rn-222. The most commonly occurring uranium isotope, U-238, has a half-life of approximately 4,500,000,000 years. In about 4.5 billion years, one-half of a quantity of U-238 is transmuted into lead, Pb-206.

Linear Models (cont’d.)

Example



EXAMPLE 2 Half-Life of Plutonium

A breeder reactor converts relatively stable uranium-238 into the isotope plutonium-239. After 15 years it is determined that 0.043% of the initial amount A_0 of the plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

$A(t)$ = amount remaining of Ur-238
$A(0) = A.$

$A(15) = (1-0.00043)A.$ = remaining after 15 years

$dA/dt = kA \rightarrow dA/dt - kA = 0,\ A(0) = A.$

$u = e^{-kt}$

$A(t) = A.\ e^{kt}$
insert $A(15) = 0.99957\ A. \rightarrow k = -2.867*10^{-5}$

for half of the Ur to disintegrate= 24180 years

Linear Models (cont'd.)

Carbon Dating About 1950, a team of scientists at the University of Chicago led by the American physical chemist **Willard F. Libby** (1908–1980) devised a method using a radioactive isotope of carbon as a means of determining the approximate ages of carbonaceous fossilized matter. The theory of carbon dating is based on the fact that the radioisotope carbon-14 is produced in the atmosphere by the action of cosmic radiation on nitrogen-14. The ratio of the amount of C-14 to the stable C-12 in the atmosphere appears to be a constant, and as a consequence the proportionate amount of the isotope present in all living organisms is the same as that in the atmosphere. When a living organism dies, the absorption of C-14, by breathing, eating, or photosynthesis, ceases. Thus by comparing the proportionate amount of C-14, say, in a fossil with the constant amount ratio found in the atmosphere, it is possible to obtain a reasonable estimation of its age. The method is based on the knowledge of the half-life of C-14. Libby's calculated value for the half-life of C-14 was approximately 5600 years and is called the **Libby half-life**. Today the commonly accepted value for the half-life of C-14 is the **Cambridge half-life** that is close to 5730 years. For his work, Libby was award the Nobel Prize for chemistry in 1960. Libby's method has been used to date wooden furniture in Egyptian tombs, the woven flax wrappings of the Dead Sea Scrolls, and the cloth of the enigmatic Shroud of Turin.

Linear Models (cont'd.)

Example

EXAMPLE 3 Age of a Fossil

A fossilized bone is found to contain 0.1% of its original amount of C-14. Determine the age of the fossil.

A(t) = amount remaining of C-14
A(0) = A.
A(t) = 0.001 A.

dA/dt = kA --> dA/dt - kA = 0, A(0) = A.

u = e^-kt

A(t) = A. e^kt
insert A(5730) =, 1/2 A. --> k = -1.2097*10^-4

for 0.001= 57103 years

Differential Equations as Mathematical Models (cont'd.)

Newton's Law of Cooling/Warming According to **Newton's empirical law of cooling**—or warming—the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, the so-called ambient temperature. If $T(t)$ represents the temperature of a body at time t, T_m the temperature of the surrounding medium, and dT/dt the rate at which the temperature of the body changes, then Newton's law of cooling/warming translates into the mathematical statement

$$\frac{dT}{dt} \propto T - T_m \quad \text{or} \quad \frac{dT}{dt} = k(T - T_m), \tag{3}$$

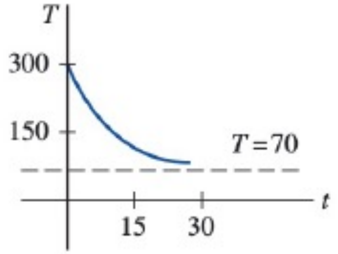
where k is a constant of proportionality. In either case, cooling or warming, if T_m is a constant, it stands to reason that $k < 0$.

Linear Models (cont'd.)

Example

EXAMPLE 4 Cooling of a Cake

When a cake is removed from an oven, its temperature is measured at 300°F. Three minutes later its temperature is 200°F. How long will it take for the cake to cool off to a room temperature of 70°F?



(a)

$T(t)$	t (in min.)
75°	20.1
74°	21.3
73°	22.8
72°	24.9
71°	28.6
70.5°	32.3

(b)

FIGURE 2.7.3 Temperature of cooling cake in Example 4

Differential Equations as Mathematical Models (cont'd.)

Mixtures The mixing of two salt solutions of differing concentrations gives rise to a first-order differential equation for the amount of salt contained in the mixture. Let us suppose that a large mixing tank initially holds 300 gallons of brine (that is, water in which a certain number of pounds of salt has been dissolved). Another brine solution is pumped into the large tank at a rate of 3 gallons per minute; the concentration of the salt in this inflow is 2 pounds of salt per gallon. When the solution in the tank is well stirred, it is pumped out at the same rate as the entering solution. See **FIGURE 1.3.3**. If $A(t)$ denotes the amount of salt (measured in pounds) in the tank at time t, then the rate at which $A(t)$ changes is a net rate:

$$\frac{dA}{dt} = \begin{pmatrix}\text{input rate}\\ \text{of salt}\end{pmatrix} - \begin{pmatrix}\text{output rate}\\ \text{of salt}\end{pmatrix} = R_{in} - R_{out}. \tag{7}$$

$$R_{in} = \overset{\substack{\text{concentration}\\ \text{of salt}\\ \text{in inflow}\\ \downarrow}}{(2\text{ lb/gal})} \cdot \overset{\substack{\text{input rate}\\ \text{of brine}\\ \downarrow}}{(3\text{ gal/min})} = \overset{\substack{\text{input rate}\\ \text{of salt}\\ \downarrow}}{(6\text{ lb/min})}.$$

$$R_{out} = \overset{\substack{\text{concentration}\\ \text{of salt}\\ \text{in outflow}\\ \downarrow}}{\left(\frac{A(t)}{300}\text{ lb/gal}\right)} \cdot \overset{\substack{\text{output rate}\\ \text{of brine}\\ \downarrow}}{(3\text{ gal/min})} = \overset{\substack{\text{output rate}\\ \text{of salt}\\ \downarrow}}{\frac{A(t)}{100}\text{ lb/min}}.$$

$$\frac{dA}{dt} + \frac{1}{100}A = 6.$$

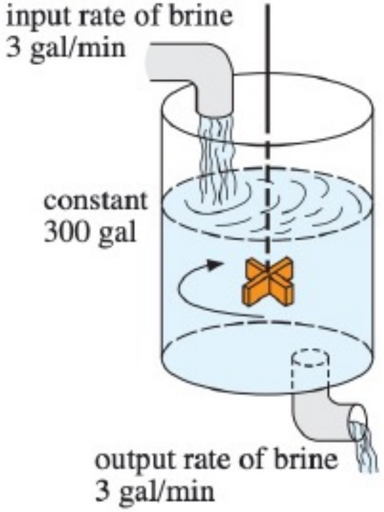


FIGURE 1.3.3 Mixing tank

Linear Models (cont'd.)

Example

EXAMPLE 5 Mixture of Two Salt Solutions

Recall that the large tank considered in Section 1.3 held 300 gallons of a brine solution. Salt was entering and leaving the tank; a brine solution was being pumped into the tank at the rate of 3 gal/min, mixed with the solution there, and then the mixture was pumped out at the rate of 3 gal/min. The concentration of the salt in the inflow, or solution entering, was 2 lb/gal, and so salt was entering the tank at the rate $R_{in} = (2\text{ lb/gal}) \cdot (3\text{ gal/min}) = 6\text{ lb/min}$ and leaving the tank at the rate $R_{out} = (x/300\text{ lb/gal}) \cdot (3\text{ gal/min}) = x/100\text{ lb/min}$. From this data and (6) we get equation (8) of Section 1.3. Let us pose the question: If there were 50 lb of salt dissolved initially in the 300 gallons, how much salt is in the tank after a long time?

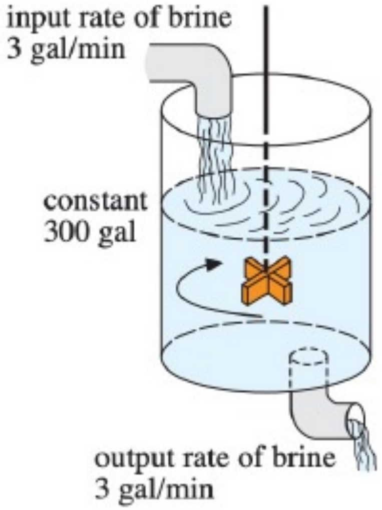
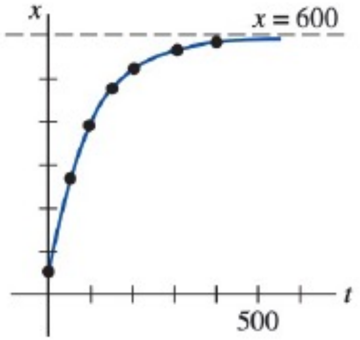


FIGURE 1.3.3 Mixing tank

Linear Models (cont'd.)

Example



(a)

t (min)	x (lb)
50	266.41
100	397.67
150	477.27
200	525.57
300	572.62
400	589.93

(b)

FIGURE 2.7.4 Pounds of salt in tank as a function of time in Example 5

SOLUTION To find the amount of salt $x(t)$ in the tank at time t, we solve the initial-value problem

$$\frac{dx}{dt} + \frac{1}{100}x = 6, \quad x(0) = 50.$$

Note here that the side condition is the initial amount of salt, $x(0) = 50$ in the tank, and not the initial amount of liquid in the tank. Now since the integrating factor of the linear differential equation is $e^{t/100}$, we can write the equation as

$$\frac{d}{dt}[e^{t/100}x] = 6e^{t/100}.$$

Integrating the last equation and solving for x gives the general solution $x(t) = 600 + ce^{-t/100}$. When $t = 0$, $x = 50$, so we find that $c = -550$. Thus the amount of salt in the tank at any time t is given by

$$x(t) = 600 - 550e^{-t/100}. \tag{7}$$

The solution (7) was used to construct the table in **FIGURE 2.7.4(b)**. Also, it can be seen from (7) and Figure 2.7.4(a) that $x(t) \to 600$ as $t \to \infty$. Of course, this is what we would expect in this case; over a long time the number of pounds of salt in the solution must be (300 gal)(2 lb/gal) = 600 lb. ≡

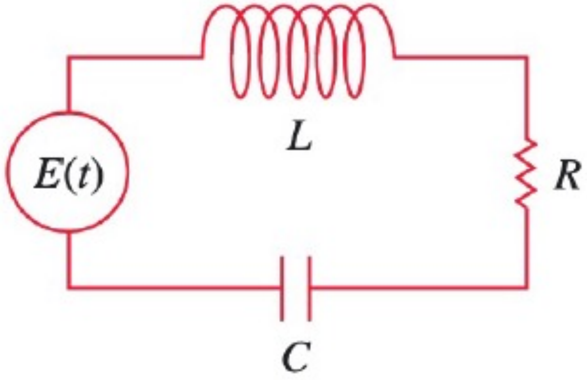
In Example 5 we assumed that the rate at which the solution was pumped in was the same as the rate at which the solution was pumped out. However, this need not be the situation; the mixed brine solution could be pumped out at a rate r_{out} faster or slower than the rate r_{in} at which the other brine solution was pumped in.

Linear Models (cont'd.)

- Example: Series Circuits
  - For a series circuit containing a resistor (*R*) and an inductor (*L*), Kirchoff's second law gives

    $L\frac{di}{dt} + Ri = E(t)$, where *E*(*t*) = impressed voltage
  - For a series circuit containing a resistor and a capacitor (*C*), Kirchoff's second law gives

    $$Ri + \frac{1}{C}q = E(t)$$
    - Since $i = dq/dt$, the linear DE $R\frac{dq}{dt} + \frac{1}{C}q = E(t)$ describes the circuit

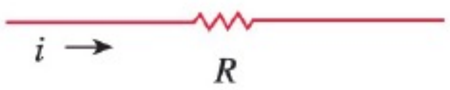


(a) *LRC*-series circuit

Resistor

resistance R: ohms (Ω)

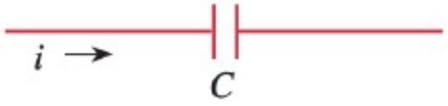
voltage drop across: iR



Capacitor

capacitance C: farads (f)

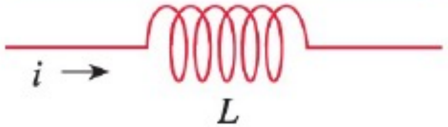
voltage drop across: $\frac{1}{C}q$



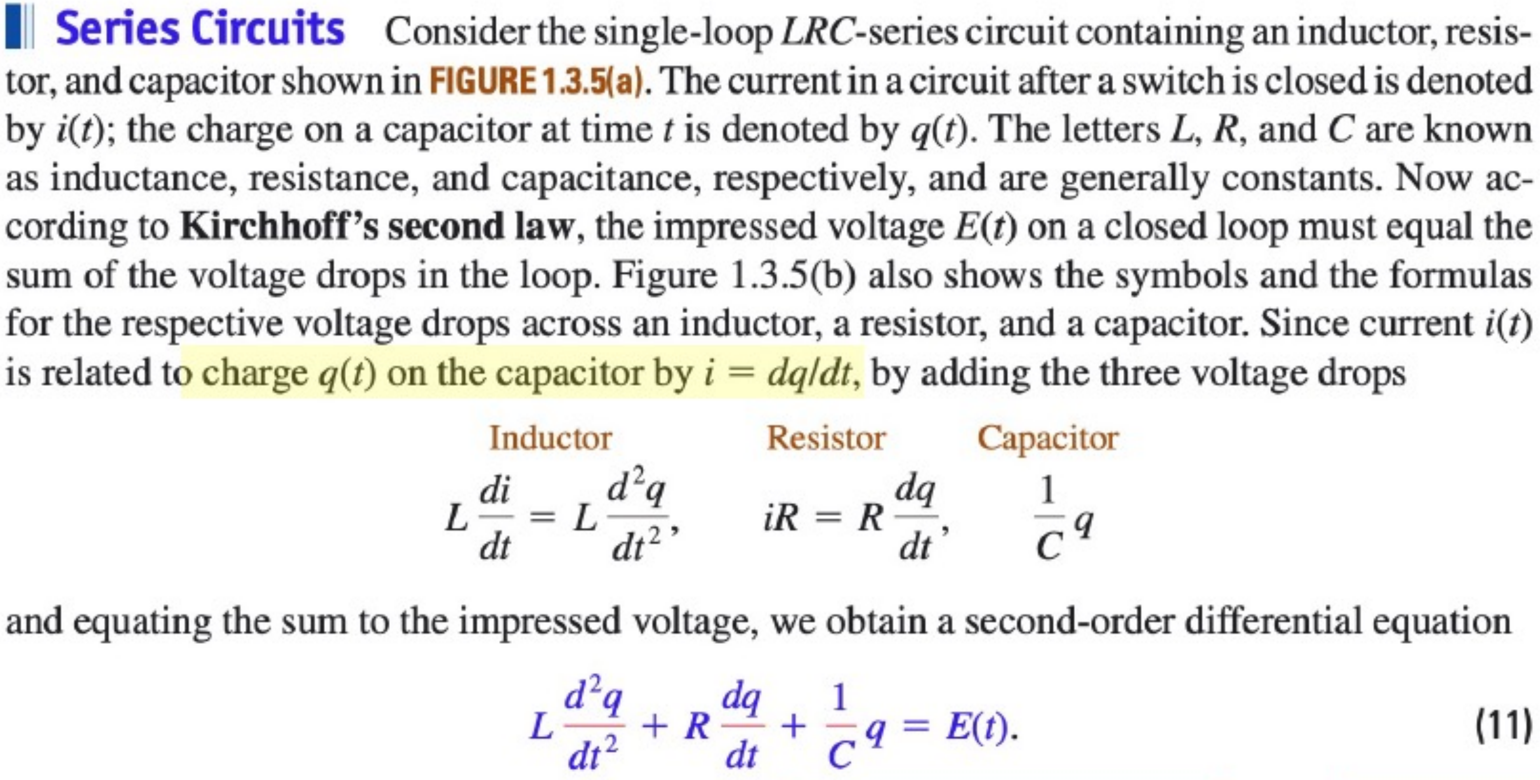
Inductor

inductance L: henrys (h)

voltage drop across: $L\frac{di}{dt}$



Linear Models (cont'd.)

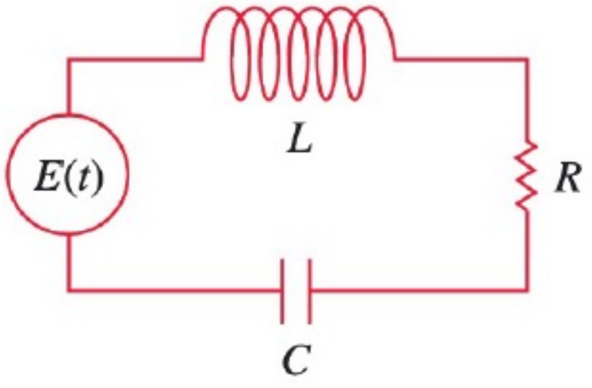


Series Circuits Consider the single-loop *LRC*-series circuit containing an inductor, resistor, and capacitor shown in **FIGURE 1.3.5(a)**. The current in a circuit after a switch is closed is denoted by $i(t)$; the charge on a capacitor at time t is denoted by $q(t)$. The letters L, R, and C are known as inductance, resistance, and capacitance, respectively, and are generally constants. Now according to **Kirchhoff's second law**, the impressed voltage $E(t)$ on a closed loop must equal the sum of the voltage drops in the loop. Figure 1.3.5(b) also shows the symbols and the formulas for the respective voltage drops across an inductor, a resistor, and a capacitor. Since current $i(t)$ is related to charge $q(t)$ on the capacitor by $i = dq/dt$, by adding the three voltage drops

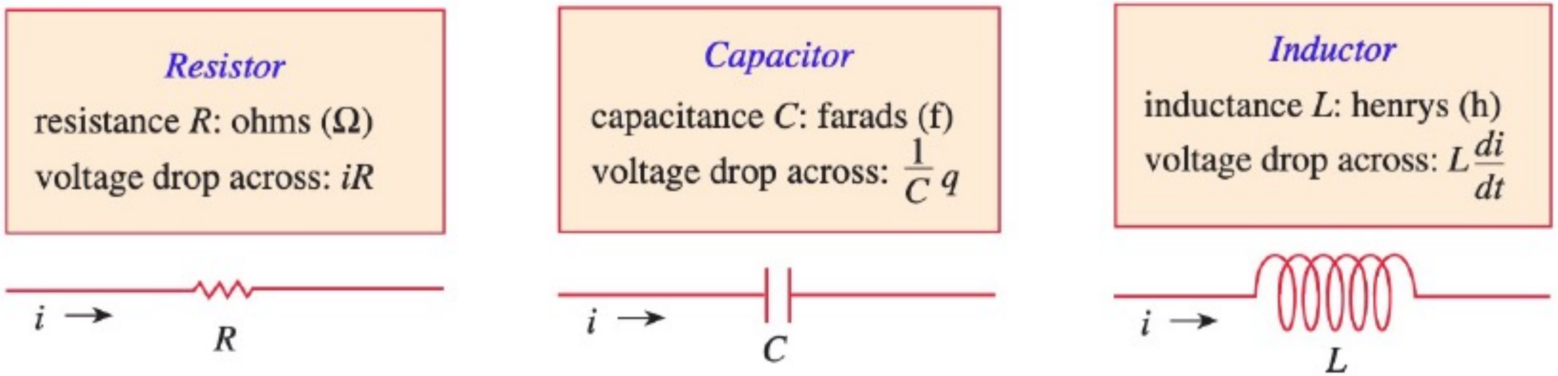
Inductor	Resistor	Capacitor
$L\dfrac{di}{dt} = L\dfrac{d^2q}{dt^2}$,	$iR = R\dfrac{dq}{dt}$,	$\dfrac{1}{C}q$

and equating the sum to the impressed voltage, we obtain a second-order differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{11}$$



(a) *LRC*-series circuit



Linear Models (cont'd.)

Series Circuits

$$L\frac{di}{dt} + Ri = E(t), \tag{8}$$

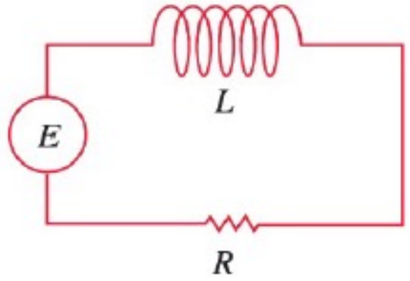


FIGURE 2.7.5 *LR*-series circuit

From (4) of Section 2.3 we can write a general solution of (8):

$$i(t) = \frac{e^{-(R/L)t}}{L}\int e^{(R/L)t}E(t)\,dt + ce^{-(R/L)t}. \tag{11}$$

In particular, when $E(t) = E_0$ is a constant, (11) becomes

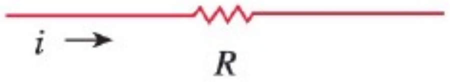
$$i(t) = \frac{E_0}{R} + ce^{-(R/L)t}. \tag{12}$$

Note that as $t \to \infty$, the second term in (12) approaches zero. Such a term is usually called a **transient term**; any remaining terms are called the **steady-state** part of the solution. In this case E_0/R is also called the **steady-state current**; for large values of time it then appears that the current in the circuit is simply governed by Ohm's law ($E = iR$).

Resistor

resistance R: ohms (Ω)

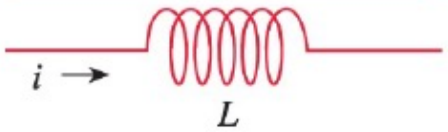
voltage drop across: iR



Inductor

inductance L: henrys (h)

voltage drop across: $L\frac{di}{dt}$



Linear Models (cont'd.)

Example

EXAMPLE 7 *LR*-Series Circuit

A 12-volt battery is connected to an *LR*-series circuit in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current i if the initial current is zero.

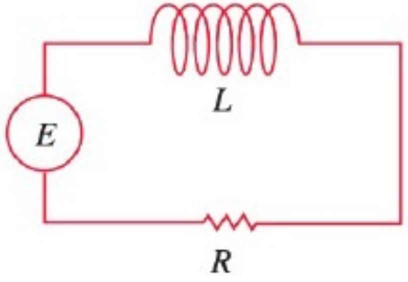
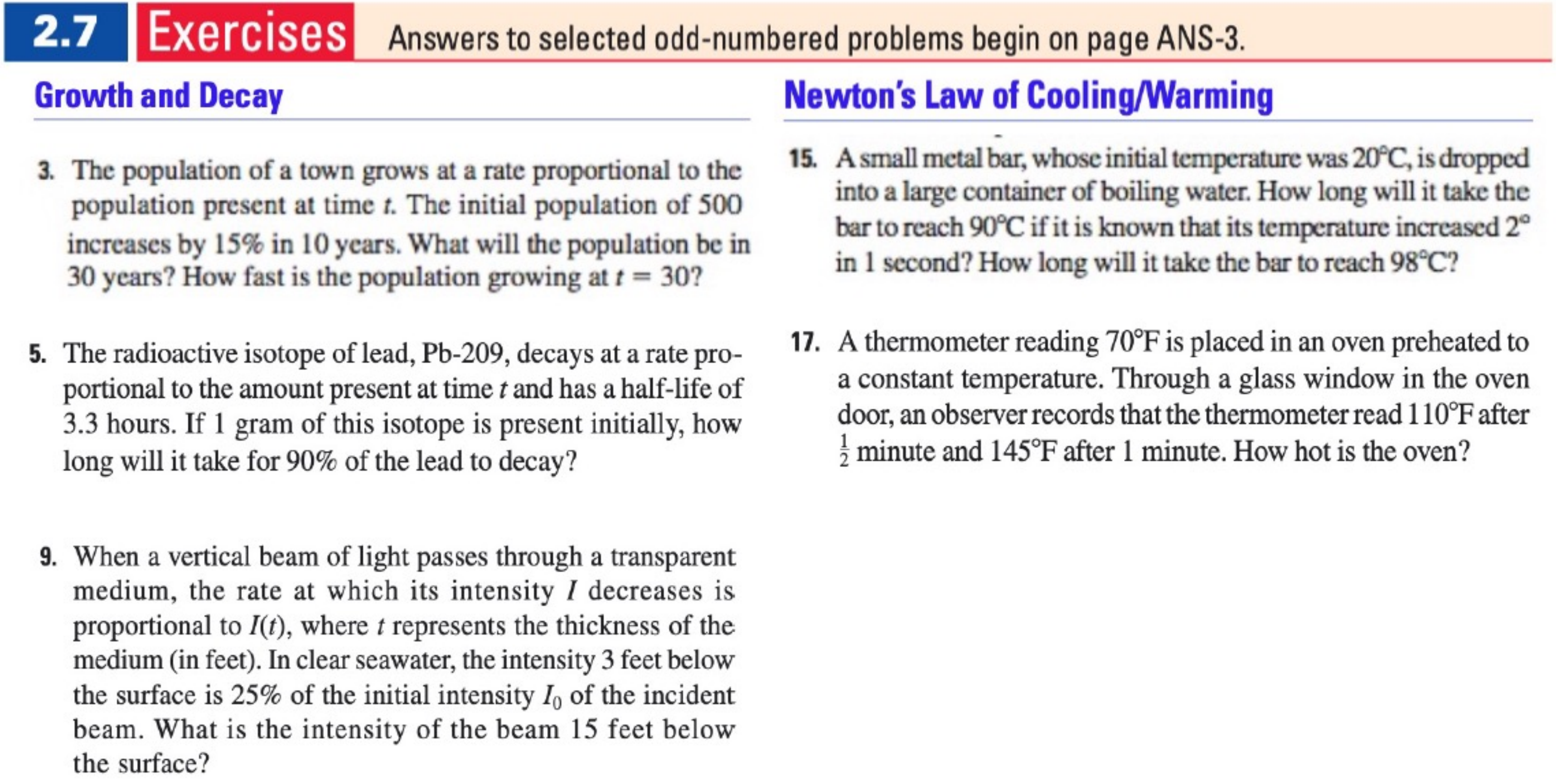


FIGURE 2.7.5 *LR*-series circuit

Modeling with Systems of First-Order DEs



2.7 Exercises

Answers to selected odd-numbered problems begin on page ANS-3.

Growth and Decay

3. The population of a town grows at a rate proportional to the population present at time t. The initial population of 500 increases by 15% in 10 years. What will the population be in 30 years? How fast is the population growing at $t = 30$?

5. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at time t and has a half-life of 3.3 hours. If 1 gram of this isotope is present initially, how long will it take for 90% of the lead to decay?

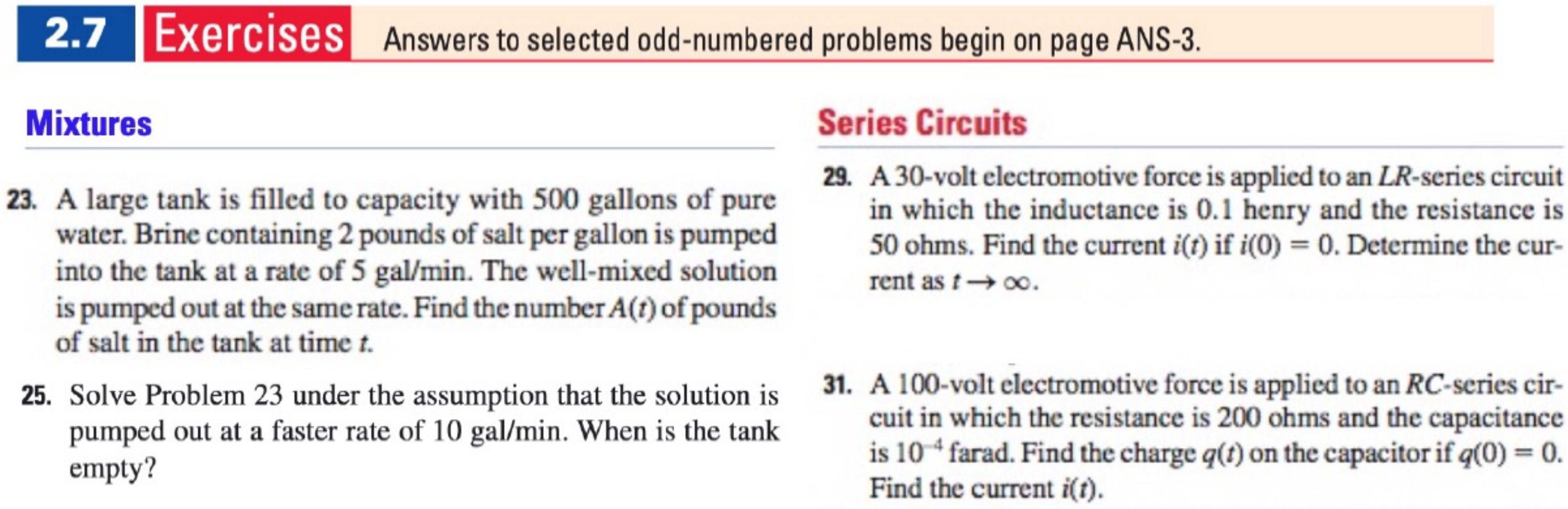
9. When a vertical beam of light passes through a transparent medium, the rate at which its intensity I decreases is proportional to $I(t)$, where t represents the thickness of the medium (in feet). In clear seawater, the intensity 3 feet below the surface is 25% of the initial intensity I_0 of the incident beam. What is the intensity of the beam 15 feet below the surface?

Newton's Law of Cooling/Warming

15. A small metal bar, whose initial temperature was 20°C, is dropped into a large container of boiling water. How long will it take the bar to reach 90°C if it is known that its temperature increased 2° in 1 second? How long will it take the bar to reach 98°C?

17. A thermometer reading 70°F is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer read 110°F after $\frac{1}{2}$ minute and 145°F after 1 minute. How hot is the oven?

Modeling with Systems of First-Order DEs



2.7 Exercises

Answers to selected odd-numbered problems begin on page ANS-3.

Mixtures

23. A large tank is filled to capacity with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 5 gal/min. The well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of pounds of salt in the tank at time t.

25. Solve Problem 23 under the assumption that the solution is pumped out at a faster rate of 10 gal/min. When is the tank empty?

Series Circuits

29. A 30-volt electromotive force is applied to an *LR*-series circuit in which the inductance is 0.1 henry and the resistance is 50 ohms. Find the current $i(t)$ if $i(0) = 0$. Determine the current as $t \to \infty$.

31. A 100-volt electromotive force is applied to an *RC*-series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge $q(t)$ on the capacitor if $q(0) = 0$. Find the current $i(t)$.