

THE UNIVERSITY OF BRITISH COLUMBIA

MATH 253

Midterm 2

14 November 2012

TIME: 50 MINUTES

FIRST NAME: Solution LAST NAME : _____

STUDENT #: _____

This Examination paper consists of 7 pages (including this one). Make sure you have all 7.

INSTRUCTIONS:

No memory aids allowed. No calculators allowed. No communication devices allowed.

PLEASE CIRCLE YOUR INSTRUCTOR'S NAME BELOW

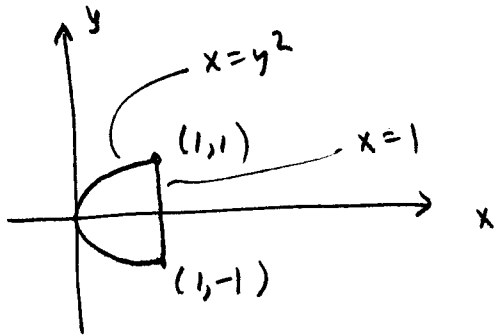
MARKING:

Q1	/10
Q2	/12
Q3	/12
Q4	/16
TOTAL	/50

NAMES OF INSTRUCTORS: Jim Bryan, Dale Peterson, Ian Hewitt, Yariv Dror-Mizrahi, Ed Richmond

Q1 [10 marks]

Find the volume of the region in 3-space which is below the surface $z = 1 + 3x^2y^2$ and lies above the region in the xy -plane enclosed by the curves $x = y^2$ and $x = 1$.



$$\text{Volume} = \int_{y=-1}^1 \int_{x=y^2}^1 (1 + 3x^2y^2) dx dy$$

$$= \int_{y=-1}^1 [x + x^3y^2]_{x=y^2}^1 dy$$

$$= \int_{y=-1}^1 (1 + \cancel{y^2} - \cancel{y^2} - y^8) dy = \left[y - \frac{1}{9} y^9 \right]_{-1}^1$$

$$= \left(1 - \frac{1}{9} \right) - \left(-1 + \frac{1}{9} \right) = 2 - \frac{2}{9} = \frac{16}{9}$$

Q2 [12 marks]

Suppose $T(x, y, z) = xy^2 - x + x^2z + yz^2$ gives the temperature at the point (x, y, z) in space.

- (a) Find an equation of the plane tangent at $(1, 2, 1)$ to the level surface of T passing through that point. [4pts]

$$\vec{\nabla} T = \langle y^2 - 1 + 2xz, 2xy + z^2, x^2 + 2yz \rangle$$

$$\vec{\nabla} T(1, 2, 1) = \langle 4 - 1 + 2, 4 + 1, 1 + 4 \rangle = \langle 5, 5, 5 \rangle$$

eqn of tangent plane: $\underline{5(x-1) + 5(y-2) + 5(z-1) = 0}$

or $\underline{(x-1) + (y-2) + (z-1) = 0}$

or $\underline{x + y + z = 4}$

- (b) At time $t = 0$, a fly passes through $(1, 2, 1)$ moving toward the point $(4, 2, 5)$ at speed of 1 unit/sec. Calculate $\frac{dT}{dt}$ at $t = 0$ for the fly. [4pts]

direction of fly: $\langle 4, 2, 5 \rangle - \langle 1, 2, 1 \rangle = \langle 3, 0, 4 \rangle$

unit vector $\vec{u} = \langle \frac{3}{5}, 0, \frac{4}{5} \rangle$

$$\left. \frac{dT}{dt} \right|_{t=0} = \vec{\nabla} T \cdot \vec{u} = \langle 5, 5, 5 \rangle \cdot \langle \frac{3}{5}, 0, \frac{4}{5} \rangle = 7$$

- (c) A worm crawling on the plane $2x - y + 2z = 2$ passes through the point $(1, 2, 1)$. The worm wishes to keep his temperature constant while increasing z . In which direction should the worm move? Express your answer as a unit vector. [4pts]

direction of worm is perpendicular to $\langle 2, -1, 2 \rangle$
 since worm crawls on the plane $2x - y + 2z = 2$

direction of worm is perpendicular to $\langle 5, 5, 5 \rangle$ since
 worm wants his temperature constant.

Thus we seek a unit vector, with positive $\vec{k}$ component,
 which is perpendicular to $\langle 2, -1, 2 \rangle$ & $\langle 5, 5, 5 \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ 5 & 5 & 5 \end{vmatrix} = 5 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 5 \langle -3, 0, 3 \rangle$$

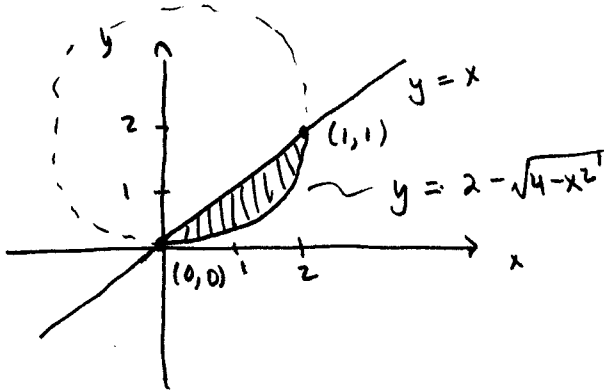
make it a unit vector $\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$

Q3 [12 marks]

Consider the following iterated integral

$$\int_0^2 \int_{2-\sqrt{4-x^2}}^x f(x,y) dy dx.$$

- (a) Sketch the region of integration. Be sure to label your axes and clearly mark x and y values on the axes. Give the coordinates of any intersection points. [4pt]



$$\begin{aligned} y &= 2 - \sqrt{4-x^2} \\ \Rightarrow (y-2)^2 &= 4-x^2 \\ \Rightarrow (y-2)^2 + x^2 &= 2^2 \quad \leftarrow \text{circle of radius 2 centered at } (0,2) \\ \Rightarrow x^2 &= 4 - (y-2)^2 \\ x &= \sqrt{4 - (y-2)^2} \end{aligned}$$

- (b) Change the order of integration to $dx dy$. [4pt]

$$\int_{y=0}^2 \int_{x=y}^{\sqrt{4-(y-2)^2}} f(x,y) dx dy$$

- (c) Convert the integral to polar coordinates. [4pt]

$$\int_{\theta=0}^{\pi/4} \int_{r=0}^{4\sin\theta} f(r\cos\theta, r\sin\theta) r dr d\theta$$

$$\begin{aligned} y^2 - 4y + x^2 &= 0 \\ y^2 + x^2 &= 4y \\ r^2 &= 4r\sin\theta \\ r &= 4\sin\theta \end{aligned}$$

Q4 [16 marks]

Consider the function $f(x, y) = (4y + 7)e^{-x^2 - y^2}$ on the domain $x^2 + y^2 \leq 1$.

(a) Find all critical points of f which are inside the domain [4 pts]

$$f_x = -2x(4y+7)e^{-x^2-y^2} \quad f_y = 4e^{-x^2-y^2} - 2y(4y+7)e^{-x^2-y^2}$$

$$f_x = 0 \Rightarrow -2x(4y+7) = 0 \quad f_y = 0 \Rightarrow 4 - 2y(4y+7) = 0$$

$$\downarrow \quad x=0 \text{ or } (4y+7=0) \text{ this doesn't work since plugging in here gives } 4=0$$

$$\text{so } x=0 \text{ and } 4 - 2y(4y+7) = 0 \Rightarrow 4 - 8y^2 - 14y = 0 \Rightarrow 8y^2 + 14y - 4 = 0$$

$$\Rightarrow (8y - 2)(y + 2) = 0 \Rightarrow y = -2 \text{ or } \frac{1}{4} \text{ critical points}$$

$$\text{are } (0, -2) \text{ and } (0, \frac{1}{4}) \text{ only } \underline{(0, \frac{1}{4})} \text{ is in the domain}$$

(b) Classify each of the critical points on the inside of the domain as a "local maximum", "local minimum", "saddle points", or "discriminant is zero". [4 pts]

$$f_{xx} = -2(4y+7)e^{-x^2-y^2} + 4x^2(4y+7)e^{-x^2-y^2}$$

$$\cancel{f_{xy} = -8ye^{-x^2-y^2} - 2x^2y e^{-x^2-y^2}} \quad f_y = (4 - 2y(4y+7))e^{-x^2-y^2} = (4 - 8y^2 - 14y)e^{-x^2-y^2}$$

$$f_{yy} = (-16y - 14)e^{-x^2-y^2} - 2y(4 - 8y^2 - 14y)e^{-x^2-y^2}$$

$$f_{xy} = -2x(4 - 8y^2 - 14y)e^{-x^2-y^2}$$

$$f_{xx}(0, \frac{1}{4}) = -2(1+7)e^{-\frac{1}{16}} = -16e^{-\frac{1}{16}}$$

$$f_{yy}(0, \frac{1}{4}) = (-4 - 14)e^{-\frac{1}{16}} - \frac{2}{4}(4 - \frac{1}{2} - \frac{7}{2})e^{-\frac{1}{16}} = -18e^{-\frac{1}{16}}$$

$$f_{xy}(0, \frac{1}{4}) = 0 \text{ so } D = f_{xx}f_{yy} - f_{xy}^2 = (-16)e^{-\frac{1}{16}}(-18)e^{-\frac{1}{16}} > 0$$

$$\text{and } f_{xx} < 0 \Rightarrow (0, \frac{1}{4}) \text{ is a } \underline{\underline{\text{local maximum}}}$$

(c) Use the method of Lagrange multipliers to find the maximum and minimum values of f on the boundary of the domain. [6pt]

$$\textcircled{1} \quad g(x,y) = x^2 + y^2 = 1 \quad f_x = \lambda g_x \quad f_y = \lambda g_y$$

$$\textcircled{2} \quad -2x(4y+7)e^{-x^2-y^2} = \lambda 2x \Rightarrow x=0 \text{ or } \lambda = -(4y+7)e^{-x^2-y^2}$$

$$\textcircled{3} \quad (4-8y^2-14y)e^{-x^2-y^2} = \lambda 2y \quad \swarrow \text{sub into } \textcircled{2}$$

$$x=0 \text{ sub into } \textcircled{1} \Rightarrow y = \pm 1$$

so $(x,y) = (0,1), (0,-1)$ are solutions

$$(4-8y^2-14y)e^{-x^2-y^2} = -2y(4y+7)e^{-x^2-y^2}$$

$$4-8y^2-14y = -8y^2-14y \Rightarrow 4=0 \quad \text{no solution}$$

so $(0,1)$ & $(0,-1)$ are the only possibilities

$$\underbrace{f(0,1) = 11e^{-1}}_{\text{max on boundary}} \quad \underbrace{f(0,-1) = 3e^{-1}}_{\text{min on boundary}}$$

(d) Find the absolute maximum and minimum values of f on its whole domain. You may use the fact that $e^{15/16} > 11/8$. [2 pts]

Abs. max/min must occur at a crit pt on the interior or a max/min on the boundary. So only possibilities are $(0,1), (0,-1), (0, \frac{1}{4})$

$$f(0,1) = 11e^{-1}$$

$$\text{since } e^{15/16} > 11/8$$

$$f(0,-1) = 3e^{-1}$$

$$\Rightarrow 8e^{1-1/16} > 11$$

$$f(0, \frac{1}{4}) = 8e^{-1/16}$$

$$\Rightarrow 8e^{-1/16} > 11e^{-1}$$

so $3e^{-1}$ is absolute min and $8e^{-1/16}$ is absolute max.