

THE UNIVERSITY OF BRITISH COLUMBIA

MATH 253

Midterm 1

10 October 2012

TIME: 50 MINUTES

FIRST NAME: Solutions LAST NAME : _____

STUDENT #: _____

This Examination paper consists of 6 pages (including this one). Make sure you have all 6.

INSTRUCTIONS:

No memory aids allowed. No calculators allowed. No communication devices allowed.

PLEASE CIRCLE YOUR INSTRUCTOR'S NAME BELOW

MARKING:

Q1	/10
Q2	/10
Q3	/10
Q4	/10
TOTAL	/40

NAMES OF INSTRUCTORS: Jim Bryan, Dale Peterson, Ian Hewitt, Yariv Dror-Mizrahi, Ed Richmond

Q1 [10 marks]

Find the partial derivatives f_x , f_y , and f_{xy} of the following functions:

(a)

$$f(x, y) = xe^{xy}$$

$$f_x = e^{xy} + xy e^{xy} = (1 + xy) e^{xy}$$

$$f_y = x^2 e^{xy} \quad f_{xy} = x e^{xy} + (1 + xy) x e^{xy} = (2x + x^2 y) e^{xy}$$

(b)

$$f(x, y) = x \sin(e^y)$$

$$f_x = \sin(e^y) \quad f_y = x e^y \cos(e^y)$$

$$f_{xy} = e^y \cos(e^y)$$

(c)

$$f(x, y) = \int_y^x t \sin(e^t) dt$$

$$f(x, y) = F(x) - F(y) \quad \text{where} \quad F'(t) = t \sin(e^t)$$

so $f_x = F'(x) = x \sin(e^x)$

$$f_y = -F'(y) = -y \sin(e^y)$$

$$f_{xy} = 0$$

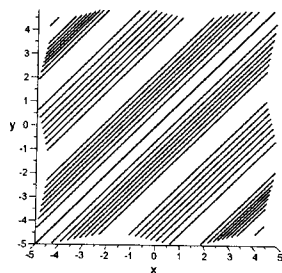
Q2 [10 marks]

Match each function with its contour plot (labeled A-I).

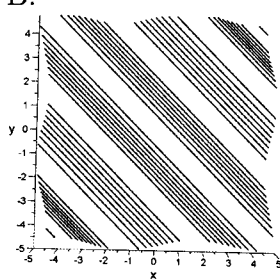
$f(x, y) = \sin(2x) + \sin(y)$ I $f(x, y) = \cos(x + y)$ B $f(x, y) = 3x - y^2$ H

$f(x, y) = (x - 2)(y + 1)$ C $f(x, y) = x^2 - y^2$ G

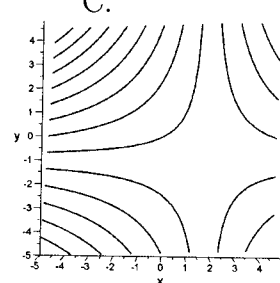
A.



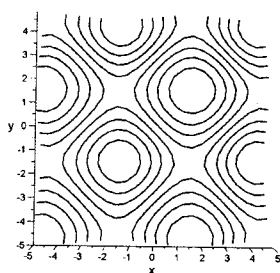
B.



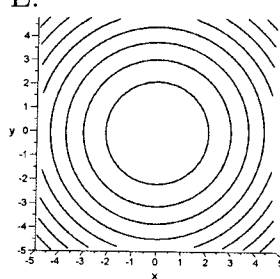
C.



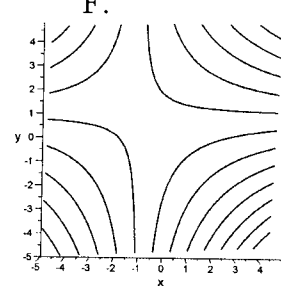
D.



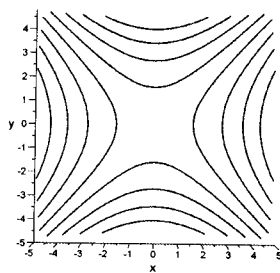
E.



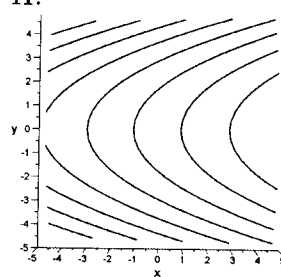
F.



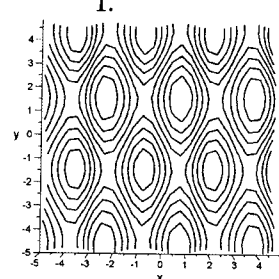
G.



H.



I.



Q3 [10 marks]

Consider the surface $z = x^2 - 6xy + 2y^3$.

$$z = z_0 + \frac{\partial z}{\partial x}(x-x_0) + \frac{\partial z}{\partial y}(y-y_0)$$

$$\frac{\partial z}{\partial x} = 2x - 6y$$

$$\frac{\partial z}{\partial y} = -6x + 6y^2$$

(a) Find an equation for the tangent plane to the surface at $(1, 2, 5)$.

$$z = 5 + (2-12)(x-1) + (-6+6 \cdot 4)(y-2)$$

$$z = 5 - 10(x-1) + 18(y-2)$$

(b) On the surface near $(1, 2, 5)$, there is a point $(x, 1.99, 5.02)$. Find an approximate value for x .

near $(1, 2, 5)$

$$z \approx 5 - 10(x-1) + 18(y-2)$$

⊗

$$\text{so } 5.02 \approx 5 - 10(x-1) + 18(1.99-2)$$

$$0.02 \approx -10(x-1) - 0.18$$

$$\Rightarrow 0.2 \approx -10(x-1) \Rightarrow x-1 \approx -0.02$$

$$x \approx 0.98$$

(c) Find all points on the surface where the tangent plane is parallel to the plane $2x+6y+z=4$.

tangent plane $0 = \frac{\partial z}{\partial x}(x-x_0) + \frac{\partial z}{\partial y}(y-y_0) - (z-z_0)$ is

normal to $\left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1 \right\rangle = \langle 2x_0 - 6y_0, -6x_0 + 6y_0^2, -1 \rangle$

which needs to be parallel to $\langle 2, 6, 1 \rangle$

$$\Rightarrow -2 = 2x_0 - 6y_0 \quad -6 = -6x_0 + 6y_0^2 \Rightarrow 1 = x_0 - y_0^2$$

$$\Rightarrow x_0 = 1 + y_0^2 \quad \text{so} \quad -2 = 2(1+y_0^2) - 6y_0 \Rightarrow 0 = 1 + 1 + y_0^2 - 3y_0$$

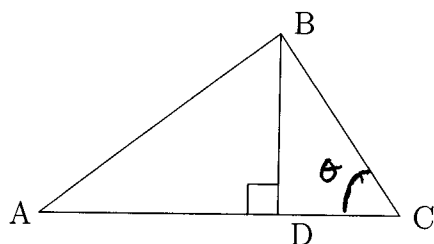
$$y_0^2 - 3y_0 + 2 = 0 \Rightarrow (y_0 - 2)(y_0 - 1) = 0 \Rightarrow y_0 = 2 \text{ or } 1 \Rightarrow x_0 = 5 \text{ or } 2$$

$$\text{so } (x_0, y_0, z_0) = (5, 2, -19) \text{ or } (2, 1, -6)$$

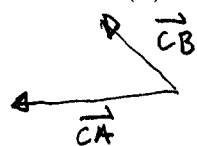
$$z_0 = 5^2 - 6 \cdot 5 \cdot 2 + 2 \cdot 8 = 25 - 60 + 16$$

Q4 [10 marks]

Consider the triangle formed by the three points $A = (4, \frac{3}{\sqrt{2}}, 0)$, $B = (0, 0, \frac{3}{\sqrt{2}})$, and $C = (-3, \frac{3}{\sqrt{2}}, 0)$. Let D be the point obtained by dropping a perpendicular line from B to the side AC as indicated in the following picture. **Please note that the angles and distances of the triangle in this drawing are not necessarily accurate.**



(a) Find the angle between the sides AC and BC .



$$\vec{CA} = \langle 4, \frac{3}{\sqrt{2}}, 0 \rangle - \langle -3, \frac{3}{\sqrt{2}}, 0 \rangle = \langle 7, 0, 0 \rangle$$

$$\vec{CB} = \langle 0, 0, \frac{3}{\sqrt{2}} \rangle - \langle -3, \frac{3}{\sqrt{2}}, 0 \rangle = \langle 3, -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}} \rangle$$

$$\vec{CA} \cdot \vec{CB} = 21 = |\vec{CA}| |\vec{CB}| \cos \theta = 7 \cdot 3 \sqrt{1 + \frac{1}{2} + \frac{1}{2}} \cos \theta$$

$$= 21 \sqrt{1 + \frac{1}{2} + \frac{1}{2}} \cos \theta \quad \text{so} \quad \cos \theta = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\theta = 45^\circ \text{ or } \pi/4}$$

(b) Find the area of the triangle ABC .

$$\text{Area} = \frac{1}{2} |\vec{CB} \times \vec{CA}| = \frac{1}{2} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 7 & 0 & 0 \end{vmatrix} = \frac{21}{2} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \frac{21}{2} \sqrt{\frac{1}{2} + \frac{1}{2}} = \boxed{\frac{21}{2}}$$

(c) Find the equation of the plane containing the points A , B , and C .

$\vec{CB} + \vec{CA} = \frac{21}{2} \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ is normal to the plane

so $\langle 0, 1, 1 \rangle$ is also normal let $(x_0, y_0, z_0) = (0, 0, \frac{3}{\sqrt{2}})$

using $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$ we get

$$y + z - \frac{3}{\sqrt{2}} = 0$$

(d) Find a unit vector which is normal to the plane.

make $\langle 0, 1, 1 \rangle$ into a unit vector:

$$\vec{N} = \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

(e) Find the coordinates of the point D .

$$\vec{CD} = \text{Proj}_{\vec{CA}} \vec{CB} = \vec{CA} \frac{\vec{CA} \cdot \vec{CB}}{|\vec{CA}|^2} = \langle 7, 0, 0 \rangle \frac{21}{7^2} = \langle 3, 0, 0 \rangle$$

The coords of D is given by $\vec{C} + \vec{CD} = \langle -3, \frac{3}{\sqrt{2}}, 0 \rangle + \langle 3, 0, 0 \rangle$
 $= \langle 0, \frac{3}{\sqrt{2}}, 0 \rangle$

$$D = (0, \frac{3}{\sqrt{2}}, 0)$$