Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 1322C Midterm Exam 2

March 16th, 2020

Professor: Guy Beaulieu

LAST NAME: Solutions **First name:** ____________

Student number: ____________

Instructions:

- You have 75 minutes to complete the exam.
- It is a test with closed books and without calculator. The use of cellphones, pagers or any other device that can transmit or store information **is not allowed.**
- Read each question carefully before answering it.
- This exam is divided into two parts:
  - Part one: It includes 6 multiple choice questions, each 2 marks. You must enter your answers in the table provided on page 2 of the exam. There will be no partial points for multiple choice questions.
  - Part two: It includes 3 development questions. The correct answer requires legible and logical written justification. You have to convince me that you know why your solution is the right one. **Clearly** frame your final responses.
- Use the space specified to answer each question. If you don't have enough space you may use the back of any other page. In this case indicate clearly where the solution continues and where is the final response.
- Do not detach the exam.
- Good Luck!

It is prohibited to use cell phones, unauthorized electronic devices or course notes (unless it is an open book exam). Phones and devices must be closed and stored in your bag: you cannot leave them in your pockets or on you. Otherwise, you may be asked to leave the room immediately and allegations of academic fraud may be made and the result may be 0 (zero) for the exam.

By signing, you acknowledge that you have complied with the above statement.

Signature: ____________

"Answers to multiple choice questions"

Q2 E	Q3 A	Q4 C
Q5 C	Q6 E	

"Do not write anything in this table"

Question	QCM	7	8	9	Total
Maximum	13	4	5	3	25
Mark					

I : MULTIPLE CHOICE QUESTIONS: 1-6 (write your final answer for questions 2-6 in the table on page 2)

QUESTION 1.

For each statement, write True or False in the provided box.

$\boxed{\text{True}}$ If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

$\boxed{\text{False}}$ If the series $\sum_{n=1}^{\infty} |a_n|$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

$\boxed{\text{False}}$ If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges by the Alternating Series Test.

QUESTION 2. The sum of the telescopic series $\sum_{n=3}^{\infty} \frac{6}{n(n-2)}$, if it exists, is:

A) $-1/2$ B) $2/3$ C) $7/6$ D) $5/2$ $\boxed{\text{E) } 9/2}$ F) ∞

$$\frac{6}{n(n-2)} = \frac{A}{n} + \frac{B}{(n-2)}$$

$$6 = A(n-2) + Bn$$

$$@ n=0 \Rightarrow 6 = -2A \Rightarrow A = -3$$

$$@ n=2 \Rightarrow 6 = 2B \Rightarrow B = 3$$

$$\therefore \frac{6}{n(n-2)} = \frac{(-3)}{n} + \frac{(3)}{(n-2)} = \frac{3}{(n-2)} - \frac{3}{n}$$

$$\therefore S_k = \sum_{n=3}^{k}\left(\frac{6}{n(n-2)}\right) = \sum_{n=3}^{k}\left(\frac{3}{(n-2)} - \frac{3}{n}\right)$$

$$= \underbrace{\left(3 - \cancel{1}\right)}_{n=3} + \underbrace{\left(\frac{3}{2} - \cancel{\frac{3}{4}}\right)}_{n=4} + \underbrace{\left(\cancel{1} - \cancel{\frac{3}{5}}\right)}_{n=5} + \underbrace{\left(\cancel{\frac{3}{4}} - \cancel{\frac{3}{6}}\right)}_{n=6} + \underbrace{\left(\cancel{\frac{3}{5}} - \cancel{\frac{3}{7}}\right)}_{n=7} + \cdots + \underbrace{\left(\cancel{\frac{3}{k-3}} - \frac{3}{k-1}\right)}_{n=k-1} + \underbrace{\left(\cancel{\frac{3}{k-2}} - \frac{3}{k}\right)}_{n=k}$$

$$= 3 + \frac{3}{2} - \frac{3}{k-1} - \frac{3}{k}$$

$$\lim S_k = \lim\left(3 + \frac{3}{2} - \frac{3}{k-1} - \frac{3}{k}\right) = 3 + 3/2 = \boxed{9/2}$$

QUESTION 3.

A series with positive terms $\sum_{n=0}^{\infty} a_n$ satisfies $0 \leq a_n \leq \frac{3^{n+1}}{4^n}$ for all $n \geq 0$. What can we conclude about this series?

A) It is convergent and $\leq \sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n} = 12$

B) It is convergent and $\leq \sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n} = 9$

C) It is convergent and $\geq \sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n} = 12$

D) It is convergent and $\geq \sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n} = 9$

E) It is divergent.

F) We cannot determine if this series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n} = \sum_{n=0}^{\infty} \frac{3 \cdot 3^n}{4^n} = \sum_{n=0}^{\infty} 3\left(\frac{3}{4}\right)^n$$ geom. series with $a = 3$ and $r = 3/4$ conv. since $|r| = \frac{3}{4} < 1$

$\therefore$ converges by Comp. Test.

$$0 \leq a_n \leq \frac{3^{n+1}}{4^n} \Rightarrow \sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n} = \frac{3}{1 - 3/4} = 12$$

$\therefore$ ~~converges~~

QUESTION 4. A tank contains 30 kg of salt dissolved in 500 L of water. Brine that contains 0.3 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. Let $Q(t)$ be the amount of salt (in kg) in the tank in the tank after t minutes. Find the differential equation satisfied by $Q(t)$.

A) $\frac{dQ}{dt} = 1 - \frac{Q}{50}$

B) $\frac{dQ}{dt} = 2 - \frac{Q}{50}$

C) $\frac{dQ}{dt} = 3 - \frac{Q}{50}$

D) $\frac{dQ}{dt} = 1 - \frac{Q}{250}$

E) $\frac{dQ}{dt} = 2 - \frac{Q}{250}$

F) $\frac{dQ}{dt} = 3 - \frac{Q}{250}$

$$\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= \left(0.3 \frac{kg}{L} \cdot 10 \frac{L}{min}\right) - \left(\frac{Q}{500} \frac{kg}{L} \cdot 10 \frac{L}{min}\right)$$

$$= 3 - \frac{Q}{50}$$

QUESTION 5.

According to the Remainder Estimate for the Integral Test, what is the minimum number of terms of the series $\sum_{n=1}^{\infty} \frac{1}{3\,n^{4/3}}$ you need to add to find its sum within 10^{-2}.

A) 999999 B) 100000 C) 1000000 D) 500000 E) 99999 F) 10000000

We know $R_n \leq \int_n^{\infty} \frac{1}{3x^{4/3}}\,dx = \lim_{t\to\infty}\left(\int_n^t \frac{x^{-4/3}}{3}\,dx\right)$

$= \lim_{t\to\infty}\left(\left[\frac{-1}{x^{1/3}}\right]_n^t\right)$

$= \lim_{t\to\infty}\left(\frac{-1}{\sqrt[3]{t}} + \frac{1}{\sqrt[3]{n}}\right)$

$= \frac{1}{\sqrt[3]{n}}.$

$\therefore$ find smallest integer such that

$\frac{1}{\sqrt[3]{n}} \leq 10^{-2} = \frac{1}{100}$

$\Rightarrow 100 \leq \sqrt[3]{n}$

$\Rightarrow 1000000 \leq n$

$\therefore \boxed{n = 1000000}$

QUESTION 6.

Solve the following initial-value problem:

$$\frac{dy}{dx} = 1 + y - x^2 - x^2 y \qquad y(0) = 1$$

A) $y = -2e^{x-x^3/3} + 1$ B) $y = -e^{x-x^3/3} + 2$ C) $y = -e^{x-x^3/3} - 1$

D) $y = x - \frac{x^3}{3} + 1$ E) $\boxed{y = 2e^{x-x^3/3} - 1}$ F) $y = x + \frac{x^3}{3} + 1$

$\frac{dy}{dx} = (1+y) - x^2(1+y)$

$\frac{dy}{dx} = (1+y)(1-x^2)$

$\int \frac{dy}{(1+y)} = \int (1-x^2)\,dx$

$\ln|1+y| = x - \frac{x^3}{3} + C$

$|1+y| = A e^{x - x^3/3}$

$1+y = A e^{x-x^3/3}$

$y = A e^{x-x^3/3} - 1$

since $y(0) = 1$

$1 = A e^{0-0^3/3} - 1$

$2 = A$

$\therefore \boxed{y = 2e^{x-x^3/3} - 1}$

B : DEVELOPMENT QUESTIONS: 7-9 (justify with full details and logically your answers)

QUESTION 7. [4 points]

Consider the power series $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n3^n}$.

(i) [2 points] Determine the power series associated to $f'(x)$ and from the series compute $f'(2)$. Make sure to simplify your answer.

$$f'(x) = \frac{d}{dx}\left[\sum_{n=1}^{\infty} \frac{x^n}{n3^n}\right] = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n\, 3^n} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{3^n}$$

$$f'(2) = \sum_{n=1}^{\infty} \frac{(2)^{n-1}}{3^n} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^n \Bigg\} \text{ geom. series } a = 1/3 \text{ and } r = \frac{2}{3}$$

conv. since $|r| = \frac{2}{3} < 1$

$$= \frac{(1/3)}{1 - (2/3)}$$

$$= 1$$

(ii) [2 points] Determine the power series associated to $\int f(x)\,dx$ and from the series deduce $\int_0^2 f(x)\,dx$.

$$\int f(x)\,dx = \int \left(\sum_{n=1}^{\infty} \frac{x^n}{n\,3^n}\right) dx = \left(\sum_{n=1}^{\infty} \frac{x^{n+1}}{n\,3^n(n+1)}\right) + C$$

$$\therefore \int_0^2 f(x)\,dx = \left[\sum_{n=1}^{\infty} \frac{x^{n+1}}{n\,3^n(n+1)}\right]_0^2$$

$$= \left(\sum_{n=1}^{\infty} \frac{(2)^{n+1}}{n\,3^n(n+1)}\right) - \underbrace{\left(\sum_{n=1}^{\infty} \frac{(0)^{n+1}}{n\,3^n(n+1)}\right)}_{=0}$$

$$= \sum_{n=1}^{\infty} \frac{2}{n(n+1)}\left(\frac{2}{3}\right)^n$$

QUESTION 8. [5 points] Determine the radius and interval of convergence for the series

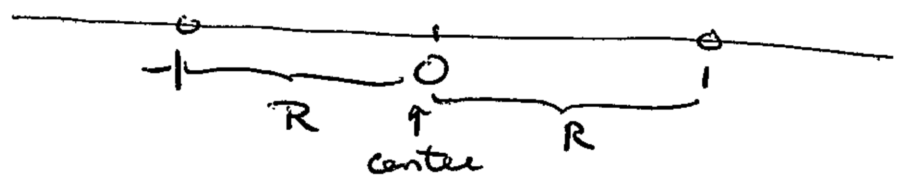
$$\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^n n x^n}{n^2+1}}_{a_n}. \quad \Bigg\} \text{ power series centered @ } x=0$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)x^{(n+1)}}{((n+1)^2+1)}\cdot\frac{(n^2+1)}{(-1)^n n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)\,x\,(n+1)(n^2+1)}{n\,(n^2+2n+2)}\right|$$

$$= |x| \lim_{n\to\infty}\left(\frac{n^3+n^2+n+1}{n^3+2n^2+2n}\right) = |x|(1) = |x|$$

$\therefore$ we need $|x|<1 \Rightarrow$ radius of conv. $R=1$



@ $x=-1$: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. Consider $b_n = \frac{1}{n}$

since 1) $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2}{n^2+1} =$

2) $\displaystyle\sum_{n=1}^{\infty} b_n$ div. (p-series $p=1$)

this series div. by Limit Comp-Test.

@ $x=1$: $\displaystyle\underbrace{\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}}_{\text{alt. series}}$ since 1) $\displaystyle\lim_{n\to\infty} b_n = 0$ and

with $b_n = \frac{n}{n^2+1}$

2) $b_{n+1} < b_n$ because (see p. 9)

QUESTION 9. [3 points] Find the first three nonzero terms of the Maclaurin series of the function $f(x) = \dfrac{x}{(1-3x)^{1/3}}$. What is its radius of convergence?

Since

$$(1+x)^k = 1 + kx + \frac{(k)(k-1)}{2!}x^2 + \frac{(k)(k-1)(k-2)}{3!}x^3 + \cdots \quad \text{with } |x|<1$$

$$(1-3x)^{-1/3} = (1+(-3x))^{-1/3}$$

$$= 1 + \left(\frac{-1}{3}\right)(-3x) + \frac{(-1/3)(-4/3)}{2!}(-3x)^2 + \frac{(-1/3)(-4/3)(-7/3)}{3!}(-3x)^3 + \cdots \quad \text{with } |-3x|<1$$

$$= 1 + x + \frac{4}{2!}x^2 + \frac{28}{3!}x^3 + \cdots \quad \text{with } |x| < \frac{1}{3}$$

~~first 3 nonzero~~ terms.

$$\therefore \frac{x}{(1-3x)^{1/3}} = x(1+(-3x))^{-1/3} = x\left(1 + x + 2x^2 + \frac{14}{3}x^3 + \cdots\right.$$

$$= \underbrace{x + x^2 + 2x^3}_{\text{first three nonzero terms}} + \frac{14}{3}x^4 + \cdots$$

radius of conv. is $R = 1/3$.

Additional page

Q8 (continued)

given $f(x) = \dfrac{x}{x^2+1}$

$D_f = [1, \infty[$

$$f'(x) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$f'(x) = 0$ or $f'(x)$ DNE

$1 - x^2 = 0$ $\underbrace{(x^2+1)^2 = 0}_{\text{impossible.}}$

$x^2 = 1$

$\underbrace{x=-1}_{\notin D_f}$ or $\underbrace{x=1}_{\in D_f}$

D_f	1	$]1, \infty[$
$f'(x)$	0	$-$

eventually decreasing.

$\therefore$ conv. by alt. series.

$\therefore$ Interval of Convergence is

$$]-1, 1].$$

Additional page