Solution to the Final Examination
MAT1320B, Fall 2019

Part I. Multiple-choice Questions ($2 \times 12 = 24$ marks)

BBAACD EEDDCF

1. The domain of the function $f(x) = \sqrt{\ln(1-x)}$ is

(A) $x < 0$; (B) $x \le 0$; (C) $x > 0$; (D) $x < 1$; (E) $x \le 1$; (F) $x > 1$.

Solution. (B) Since $\ln(1-x)$ is under the square root, $\ln(1-x) \ge 0$. This means $1-x \ge 1$, or $x \le 0$.

2. The derivative of the function $f(x) = \dfrac{\tan(x^2)}{x}$ at $x = \sqrt{\dfrac{\pi}{4}}$ is

(A) $\dfrac{4}{\pi}$; (B) $4-\dfrac{4}{\pi}$; (C) $8-\dfrac{4}{\pi}$; (D) $2-\dfrac{4}{\pi}$; (E) $4-\dfrac{2}{\pi}$; (F) $\dfrac{2}{\pi}$.

Solution. (B) By the quotient rule, $f'(x) = \dfrac{2x\sec^2(x^2)-\tan(x^2)}{x^2}$. When $x = \sqrt{\dfrac{\pi}{4}}$, $\sec^2\left(\dfrac{\pi}{4}\right) = 2$,

$\tan\left(\dfrac{\pi}{4}\right) = 1$. Hence, $\dfrac{2\times\frac{\pi}{4}\times 2-1}{\frac{\pi}{4}} = 4-\dfrac{4}{\pi}$.

3. Let $f(x) = \dfrac{(x^4+7)^{2/3}e^{x^2-1}}{\sqrt{x^2+3}}$. Found by logarithmic differentiation, $f'(1) =$

(A) $\dfrac{25}{6}$; (B) $\dfrac{25}{12}$; (C) $\dfrac{13}{6}$; (D) $\dfrac{13}{12}$; (E) $\dfrac{5}{6}$; (F) $\dfrac{5}{12}$.

Solution. (A) Taking the logarithm on both sides,

$$\ln f(x) = \frac{2}{3}\ln(x^4+7)+(x^2-1)-\frac{1}{2}\ln(x^2+3).$$

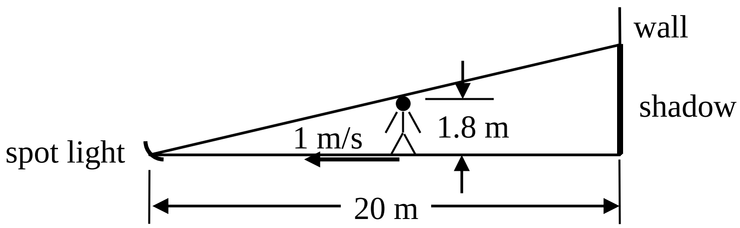
Then take the derivative with respect to x on both sides: $\dfrac{f'(x)}{f(x)} = \dfrac{8x^3}{3(x^4+7)}+2x-\dfrac{x}{x^2+3}$.

$$f'(x) = \frac{(x^4+7)^{2/3}e^{x^2-1}}{\sqrt{x^2+3}}\left(\frac{8x^3}{3(x^4+7)}-2x-\frac{x}{x^2+3}\right).$$

$$f'(1) = \frac{4}{2}\left(\frac{8}{3\times 8}+2-\frac{1}{4}\right) = 2\left(\frac{1}{3}+2-\frac{1}{4}\right) = \frac{25}{6}.$$

4. A spot light on the ground is shining on a vertical wall 20 meters away. A man 1.8 meters tall is walking from the wall towards the spot light at a speed 1 meter per second. What is the rate of change (in meters per second) of the length of his shadow on the wall when the man is 10 meters away from the spot light?

(A) 0.36; (B) 0.48; (C) 0.54; (D) 0.64; (E) 0.78; (F) 1.08.



Solution. (A) Let the distance between the man and the spot light be x, and let the length of the shadow be y. Then $x(t)$ and $y(t)$ are functions of time t. By the property of similar triangles, $\frac{y}{1.8} = \frac{20}{x}$. Taking the derivative on both sides with respect to t:

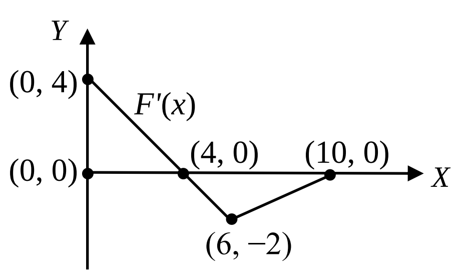
$$\frac{y'}{1.8} = -\frac{20x'}{x^2},\ y' = -\frac{36x'}{x^2}.$$

When $x = 10$ and $x' = -1$, $y' = -\frac{36 \times (-1)}{100} = 0.36$.

The length of the shadow is increasing at a rate 0.36 m/sec.

5. Let $F(x)$ be a function defined and continuous on interval [0, 10] with $F(0) = 1$. If the graph of the derivative $F'(x)$ of $F(x)$ consists of two straight line segments as given in the following figure, then $F(10)$ =

(A) 1; (B) 2; (C) 3; (D) 4; (E) 5; (F) 6.



Solution. (C) By the Net Change Theorem, $F(10) - F(0) = \int_0^{10} F'(x)dx$, which is the area of the triangle with vertices (0, 0), (0, 4), and (4, 0) minus the area of the triangle with vertices (4, 0), (6, −2), and (10, 0). Hence, $\int_0^{10} f'(x)dx = \frac{1}{2}(4\times 4) - \frac{1}{2}(2\times 6) = 2$, and $f(10) = f(0) + 2 = 3$.

6. Suppose $\int_a^b f(x)dx = x$, $\int_c^d f(x)dx = y$, $\int_a^d f(x)dx = z$. Then $\int_b^c f(x)dx =$

(A) $x + y + z$; (B) $x + y - z$; (C) $x - y + z$;
(D) $z - x - y$; (E) $-x - y - z$; (F) $z - x + y$.

Solution. (D) $\int_b^c f(x)dx = \int_b^d f(x)dx - \int_c^d f(x)dx = \left(\int_a^d f(x)dx - \int_a^b f(x)dx\right) - \int_c^d f(x)dx$
$= z - x - y$.

7. Let $f(x) = \int_0^{\arctan x} \tan t dt$. Then $f'(x) =$

(A) $\frac{x^2}{1+x^2}$; (B) $\frac{x^2}{(1+x^2)^2}$; (C) $\frac{x^2}{\sqrt{1+x^2}}$; (D) $\frac{x}{\sqrt{1+x^2}}$; (E) $\frac{x}{1+x^2}$; (F) $\frac{x}{(1+x^2)^2}$.

Solution. (E) $f'(x) = (\arctan x)' \tan(\arctan x) = \frac{x}{1+x^2}$.

8. Some values of a function $y = f(x)$ is given in the following table:

x	0	0.5	1	1.5	2
$f(x)$	1	2	4	6	7

If Simpson's rule with $n = 4$ and the given data are used to estimate definite integral $\int_0^2 f(x)dx$, which one of the following numbers is closest to this estimate?

(A) 4; (B) 5; (C) 6; (D) 7; (E) 8; (F) 9.

Solution. (E) With given data and Simpson's rule, $h = 0.5$ and

$$\int_0^2 f(x)dx \approx \frac{0.5}{3}(1 + 4 \times 2 + 2 \times 4 + 4 \times 6 + 7) = 8.$$

9. $\int_0^{\pi/2} \sin^3 x dx =$

(A) 1; (B) $\frac{1}{3}$; (C) $\frac{1}{2}$; (D) $\frac{2}{3}$; (E) $\frac{3}{2}$; (F) $\frac{4}{3}$.

Solution. (D) Let $u = \cos x$. Then $u' = -\sin x$.

$$\int_0^{\pi/2} \sin^3 x dx = -\int_1^0 (1-u^2)du = -\left[u - \frac{1}{3}u^3\right]_{u=1}^0 = \left(1 - \frac{1}{3}\right) = \frac{2}{3}.$$

10. Which one of the following statement is always true?

(A) If a function $f(x)$ is continuous at $x = a$, then $f(x)$ is differentiable at $x = a$.
(B) If a function is continuous in an interval, then it attains an absolute maximum in this interval.
(C) If $f(x)$ defined in a closed interval attains an absolute maximum at $x = a$, then $f'(a) = 0$.
(D) If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.
(E) If $f''(a) = 0$, then $f(x)$ has an inflection point at $x = a$.
(F) If $x = a$ is a critical number of function $f(x)$, then $f(x)$ attains a local maximum or a local minimum at $x = a$.

Answer. (D)

11. Consider function $f(x) = x^3 - 3x^2 + 1$ defined on a closed interval $-1 \le x \le 1$. Which one of the following statements is true?

(A) This function attains the absolute maximum at $x = -1$, and it attains the absolute minimum at $x = 0$.
(B) This function attains the absolute maximum at $x = 0$, and it attains the absolute minimum at $x = 1$.
(C) This function attains the absolute maximum at $x = 0$, and it attains the absolute minimum at $x = -1$.
(D) This function attains the absolute maximum at $x = -1$, and it attains the absolute minimum at $x = 1$.
(E) This function attains the absolute maximum at $x = 0$, and it does not have a absolute minimum.
(F) This function attains the absolute minimum at $x = -1$, and it does not attains a absolute maximum.

Solution. (C) $f'(x) = 3x^2 - 6x$. Let $f'(x) = 0$. $x = 0$ is the only critical point in its domain. When $x < 0$, $f'(x) > 0$, and, when $x > 0$, $f'(x) < 0$. This function attains a local maximum at $x = 0$. $f(0) = 1$. At the ends of its domain, $f(-1) = -3$, and $f(1) = -1$. $f(x)$ attains the absolute maximum at $x = 0$, and it attains the absolute minimum at $x = -1$.

12. $\lim_{x\to 0} (1-2x)^{1/x} =$

(A) 0; (B) 1; (C) $\sqrt{e}$; (D) e; (E) e^{-1}; (F) e^{-2}.

Solution. (F) Let $y = (1-2x)^{1/x}$. Then $\ln y = \dfrac{\ln(1-2x)}{x}$. $\lim_{x\to 0} \ln y = \lim_{x\to 0} \dfrac{\ln(1-2x)}{x} = \lim_{x\to 0} \dfrac{-2}{1-2x} = -2$.
Hence, $\lim_{x\to 0} (1-2x)^{1/x} = e^{-2}$.

Part II. Long Answer Questions (26 marks)

Write your solution in a clear and orderly manner. Your presentation counts!

1. ($4 \times 3 = 12$ marks) Find the following definite/indefinite integrals:

(a) $\int_1^e x \ln x\, dx =$

Solution. Use integration by parts. Let $u = \ln x$, and $v' = x$. Then $u' = x^{-1}$ and $v = \dfrac{1}{2}x^2$.

$$\int_1^e x\ln x dx = \frac{1}{2}\left[x^2\ln x\right]_{x=1}^e - \frac{1}{2}\int_1^e x dx = \frac{1}{2}e^2 - \frac{1}{4}\left[x^2\right]_{x=1}^e = \frac{1}{2}e^2 - \frac{1}{4}\left(e^2-1\right) = \frac{1}{4}\left(e^2+1\right).$$

(b) $\int \frac{1}{(1-x^2)^{3/2}} dx$.

Solution. Let $x = \sin u$. Then $(1 - x^2)^{3/2} = \cos^3 u$.

$$\int \frac{1}{(1-x^2)^{3/2}} dx = \int \frac{1}{\cos^3 u}(\cos u) du = \int \frac{1}{\cos^2 u} du = \tan u + C = \frac{x}{\sqrt{1-x^2}} + C .$$

(c) $\int \frac{1-2x}{x^2(1-x)} dx$.

Solution. Let $\frac{1-2x}{x^2(1-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x} = \frac{Ax(1-x)+B(1-x)+Cx^2}{x^2(1-x)}$. Then $Ax(1-x) + B(1-x) + Cx^2 = 1 - 2x$. Let $x = 0$. $B = 1$. Let $x = 1$. $C = -1$. Compare the coefficients of x^2 on both sides. $-A + C = 0$. $A = C = -1$. Hence,

$$\int \frac{1-2x}{x^2(1-x)} dx = -\int \frac{1}{x} dx + \int \frac{1}{x^2} dx - \int \frac{1}{1-x} dx = -\ln|x| - \frac{1}{x} + \ln|1-x| + C .$$

2. (8 marks) Consider the function $f(x) = \frac{\ln x + 1}{x}$ defined for $x > 0$. The first and the second derivatives of $f(x)$ are $f'(x) = -\frac{\ln x}{x^2}$, and $f''(x) = -\frac{1-2\ln x}{x^3}$.

You may use $e \approx \frac{8}{3} \approx 2.7$, and $\sqrt{e} \approx \frac{5}{3} \approx 1.65$.

(a) (1 mark) Function $f(x)$ is increasing in interval(s) $(0, 1)$.

Justify your answer:

Let $f'(x) = 0$. Then $\ln x = 0$, and $x = 1$. When $0 < x < 1$, $f'(x) > 0$, and, when $x > 1$, $f'(x) < 0$. Hence, this function is increasing in interval $0 < x < 1$, and it attains a local maximum at $x = 1$.

(b) (1 mark) The graph of $f(x)$ is concave up in interval(s) $(\sqrt{e}, \infty)$.

Justify your answer:

Let $f''(x) = 0$. Then $1 - 2\ln x = 0$, $\ln x = \frac{1}{2}$, and $x = \sqrt{e}$.

When $0 < x < \sqrt{e}$, $f''(x) < 0$, and, when $x > \sqrt{e}$, $f''(x) > 0$. The graph of $f(x)$ is concave down in interval $(0, \sqrt{e})$, and it is concave up in interval $(\sqrt{e}, \infty)$. The graph of this function has an inflection point at $\left(\sqrt{e}, \frac{3}{2\sqrt{e}}\right) \approx \left(\frac{8}{3}, \frac{10}{9}\right)$.

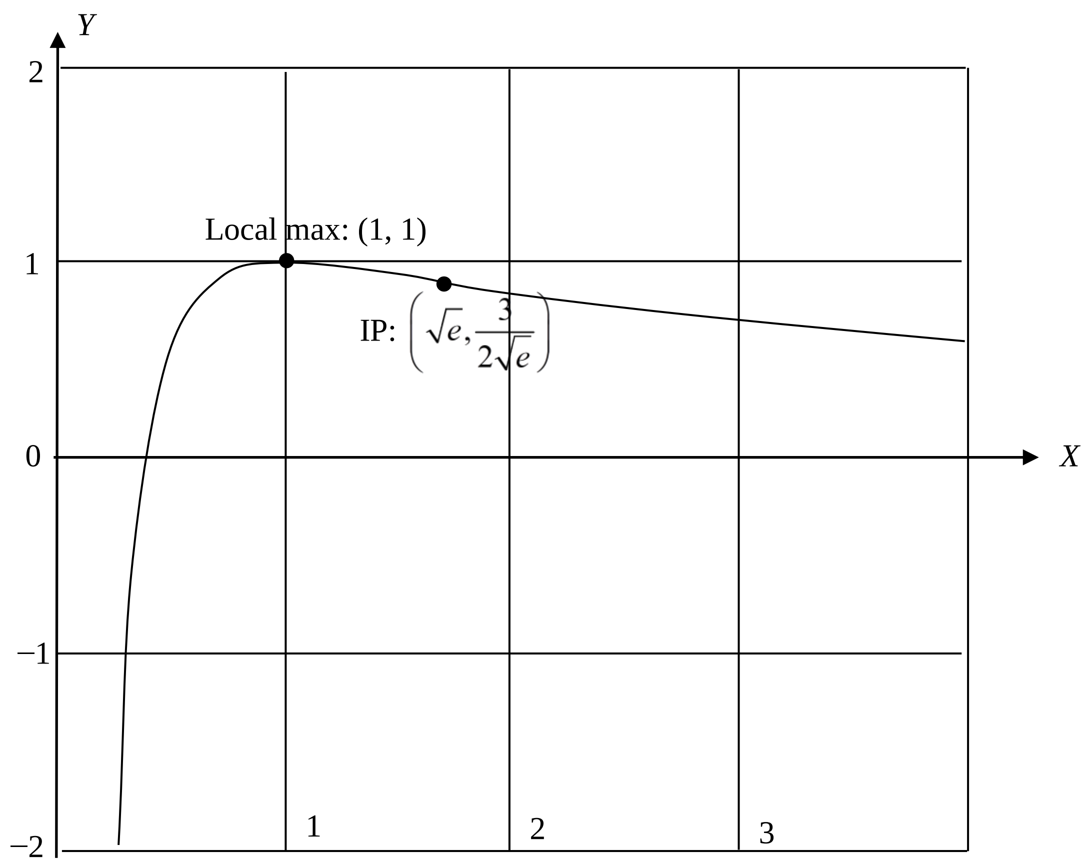
(c) (2 mark) The graph of $f(x)$ has vertical asymptote $\underline{x = 0}$, and horizontal asymptote $\underline{y = 0}$.

Justify your answer:

When x approaches 0^+, since the numerator approaches $-\infty$, and the denominator approaches 0^+, $\lim_{x \to 0^+} f(x) = -\infty$. The graph of $f(x)$ has a vertical asymptote $x = 0$.

Use L'Hopital's rule: $\lim_{x\to\infty} \frac{\ln x + 1}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$. The graph of $f(x)$ has a horizontal asymptote $y = 0$.

(d) (4 marks) Sketch the graph of $f(x)$. Mark the local extrema and inflection points, if any.



3. (6 marks) A company is to design a rectangular box of capacity 12 liters (i.e., 12000 cm^3) with a square base. The material to make the square top and the square bottom costs 0.3 cent/cm^2, and material to make the four vertical rectangle sides cost 0.2 cent/cm^2. Find the minimum cost of the material and the

dimensions of the box so that the total cost of the material is minimized. Justify what you obtained is the absolute minimum.

Remember to define all symbols that you use in your solution.

Solution. Let x be the side length of the base, and let h be the height of the box. Then the total cost of the material is $C = 0.3 \times (2x^2) + 0.2 \times (4xh) = 0.6x^2 + 0.8xh$.

Since the capacity $V = x^2h = 12000$, $h = \dfrac{12000}{x^2}$, and $C = 0.6x^2 + \dfrac{9600}{x}$. The domain of this function is $x > 0$.

Let $C' = 1.2x - \dfrac{9600}{x^2} = 0$. $x^3 = 8000$. $x = 20$. When $x = 20$, $h = 30$, and $C = 240 + 480 = 720$ cents.

Since $C' < 0$ when $x < 20$, and $C' > 0$ when $x > 20$, function C attains a local minimum at $x = 20$.

Since $\lim_{x \to 0^+} C(x) = \lim_{x \to \infty} C(x) = \infty$, $C = 720$ is the absolute minimum.