V1

1. Which of the following are subspaces of $\mathbf{R}^3$?

$U = \{(x,\ y,\ z) \in \mathbf{R}^3 \mid x - 2y + z = 0\}$ is a plane through 0 ∴ is a S.S.
$V = \{(x,\ xy,\ y) \in \mathbf{R}^3 \mid x, y \in \mathbf{R}\}$
$W = \{(x,\ y,\ z) \in \mathbf{R}^3 \mid 2x - 5z = 0\}$ is a plane through 0 ∴ is a S.S.
$X = \{(x + y, y, x - 2y) \mid x, y \in \mathbf{R}\}$ = Span {(1,0,1), (1,1,-2} ∴ is a S.S

A. Only U and V
B. Only U and W
C. Only W and X
D. Only U, V and W
E. Only U, V and X
F. Only U, W and X

– Moreover "V" dos not occur in any answer with U, W & X ∴ F is correct

– or: (0,0,1) and (1,0,0) are in V but their sum (1,0,1) is not.

2. It is known that a subspace Y of $\mathbf{R}^{110}$ can be spanned by 96 vectors, and that Y has a linearly independent set with 71 vectors. Then it is always true that:

A. $\dim Y < 71$

B. $\dim Y > 71$

C. $71 < \dim Y \leq 96$

D. $71 \leq \dim Y < 96$

E. $71 \leq \dim Y \leq 96$

F. None of the above is true.

We know from class that

the size of any l.i. set in Y ≤ dim Y ≤ the size of any spanning set for Y

∴ 71 ≤ dim Y ≤ 96

3. Suppose $\{u, v\}$ is a linearly **independent** set in vector space V, and that $w \in V$ is chosen so that $\{u, v, w\}$ is linearly **independent**. Which of the following statements is **ALWAYS** true?

A. $\{u, w\}$ is linearly dependent.

B. $\{v, w\}$ is linearly dependent.

C. $\{u, v\}$ is linearly dependent.

D. $u \in \text{span}\{v, w\}$.

E. $v \in \text{span}\{u, w\}$.

F. $w \notin \text{span}\{u, v\}$.

We know from class that if $\{u, v\}$ is l.i., and w is any vector in V, then $\{u, v, w\}$ is l.i. $\Leftrightarrow$ $w \notin \text{span}\{u, v\}$

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4. Recall the vector space $\mathcal{P}_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbf{R}\}$ of polynomial functions of degree at most 2, and define

$$W = \{p \in \mathcal{P}_2 \mid p(3) = 0\}.$$

a) Show that $W = \text{span}\{x-3, x^2-3x\}$. *(Hint: recall the Factor Theorem: if p is any polynomial and $p(a) = 0$ for some $a \in \mathbf{R}$, then $p(x) = (x-a)q(x)$ for some polynomial q of degree one less than that of p.)*

b) Explain why W is a subspace of $\mathcal{P}_2$ *without using the subspace test.*

c) Find a basis for W. (You may use without proof the fact proved in class that $\{1, x, x^2\}$ is linearly independent.)

d) Find $\dim W$.

(Remember that you must justify your answers.)

a) If $p \in W$, then by the factor theorem, $(x-3)$ is a factor of p.
Hence $W = \{p \in \mathcal{P}_2 \mid p(3)\} = \{(x-3)(a+bx) \mid a, b \in \mathbb{R}\}$
$= \{a(x-3) + b(x^2-3x) \mid a, b \in \mathbb{R}\}$
$= \text{span}\{x-3, x^2-3x\}$.

b) Since $W = \text{span}\{x-3, x^2-3x\}$, we know W is a subspace of $\mathcal{P}_2$.
("All spans are subspaces")

c) We claim $\{x-3, x^2-3x\}$ is a basis for W. By (a), it remains to show that $\{x-3, x^2-3x\}$ is l.i.. So suppose
$a(x-3) + b(x^2-3x) = 0$ for all $x \in \mathbb{R}$.
then $-3a + (a-3b)x + bx^2 = 0$ "
Since $\{1, x, x^2\}$ is l.i., this implies $-3a = 0 = a-3b = b$; in particular $a = b = 0$. Hence $\{x-3, x^2-3x\}$ is l.i.. By (a), this makes $\{x-3, x^2-3x\}$ a basis for W.

d) Since there are 2 vectors in the basis $\{x-3, x^2-3x\}$ (of W), $\dim W = 2$.

5. Let $\mathbf{M}_{22}$ denote the vector space of 2 by 2 matrices with real entries, and define

$$U = \left\{ \begin{bmatrix} a & b \\ a+b & c \end{bmatrix} \in \mathbf{M}_{22} \mid a, b, c \in \mathbf{R} \right\}.$$

note: $\begin{bmatrix} a & b \\ a+b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. (*)

a) <u>Either</u> check that U is closed under addition, <u>or</u> express U in another form so you can simply state a theorem that guarantees that U is a subspace.

(For parts (b) and (c) you may assume that U is a subspace of $\mathbf{M}_{22}$.)

b) Find a basis for U, and hence find $\dim U$.

c) Give a basis for U different from the one you gave in (b).

(Remember that you must justify your answers.)

a) By (*), $U = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and so U is a subspace of M_{22}. (In what follows, let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.)

b) By (a), $\{A, B, C\}$ spans U. Suppose $aA + bB + cC = 0$ for some scalars a, b, c. Then $\begin{bmatrix} a & b \\ a+b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which in particular implies $a = b = c = 0$. (as well as $a+b=0$). Hence $\{A, B, C\}$ is l.i., and by (a) is thus a basis for U. Thus $\dim U = 3$ (3 vectors in the basis).

c) I claim $\{2A, 2B, 2C\}$ is another basis: Let $u \in U$. Since $\{A, B, C\}$ is a basis, $u = aA + bB + cC$ for some $a, b, c \in \mathbb{R}$. Then $u = \left(\frac{a}{2}\right)2A + \left(\frac{b}{2}\right)2B + \left(\frac{c}{2}\right)2C$, so $\{2A, 2B, 2C\}$ spans U. Moreover if $a_1(2A) + a_2(2B) + a_3(2C) = 0$, then $2a_1A + 2a_2B + 2a_3C = 0$. But $\{A, B, C\}$ is l.i. so $2a_1 = 2a_2 = 2a_3 = 0$. Therefore $a_1 = a_2 = a_3 = 0$ & so $\{2A, 2B, 2C\}$ is a

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6. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you must give an explicit example - with numbers, matrices, or functions, as is appropriate!

- If you say the statement is always true, you must give a clear explanation.

a) $X = \{f \in \mathbf{F}(\mathbf{R}) \mid f(x) \geq -1 \quad \text{for all } x \in \mathbf{R}\}$ is a subspace of $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$.

Define $f(x) = 2$, for all $x \in \mathbb{R}$. Then $f \in X$, but $(-1)f \notin X$, since $(-1)f(x) = -2 \not\geq -1$ (for all $x \in \mathbb{R}$!)
Hence X is not closed under mult. by scalars and so is not a subspace

ANSWER $\boxed{\text{False}}$

b) If V is a vector space and $\{v_1, v_2, v_3\} \subset V$ is linearly independent, then $\{v_1, v_2\}$ is also linearly independent.

This follows immediately from a theorem (FACT) in class: any subset of a l.i. set is l.i.

PF: Suppose $av_1 + bv_2 = 0$. Then $av_1 + bv_2 + 0v_3 = 0$. Since $\{v_1, v_2, v_3\}$ is l.i., this implies $a = b = 0 = 0$. i.e. $a = b = 0$. So $\{v_1, v_2\}$ is l.i.. $\lrcorner$

ANSWER $\boxed{\text{True}}$

6 (cont.).

c) $Y = \left\{ \begin{bmatrix} a & a \\ b & c \end{bmatrix} \in \mathbf{M}_{2\,2} \mid a, b, c \in \mathbf{R} \right\}$ is a subspace of $\mathbf{M}_{2\,2}$ of dimension 3.

Note: $Y = \operatorname{span}\left\{ \underset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{B}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{C}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$ and hence is a subspace of M_{22}. Its dimension is 3 since $aA + bB + cC = 0 \Rightarrow \begin{bmatrix} a & a \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\Leftrightarrow a = b = c = 0$, and so $\{A, B, C\}$ is a basis of Y.

ANSWER $\boxed{\text{True}}$

d) If v_1, v_2, v_3 and v_4 are non-zero vectors in a vector space V, and $U = \operatorname{span}\{v_1, v_2, v_3, v_4\}$ then $\dim U = 4$.

Let $V = \mathbb{R}^2$, $v_1 = v_2 = v_3 = v_4 = (1,0)$. Then $U = \operatorname{span}\{(1,0)\}$ has dimension 1.

ANSWER $\boxed{\text{False}}$