V1

1. Find the derivative of

$$f(x) = \frac{1}{3x+1}$$

using the definition. *You may not use any of the differentiation rules from class, only the definition.*

[2pts]

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{3(x+h)+1} - \frac{1}{3x+1}}{h}$$

$$= \lim_{h\to 0} \frac{(3x+1) - (3x+3h+1)}{(3x+3h+1)(3x+1)h}$$

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{(3x+3h+1)(3x+1)\cancel{h}}$$

$$= \frac{-3}{(3x+1)^2}$$

V1

2. Determine all values of a such that the function

$$g(x) = \begin{cases} (x-1)^2 + a & \text{if } x < 2 \\ 2^x + ax & \text{if } x \geq 2 \end{cases}$$

[1pts] is continuous everywhere. For g to be continuous at 2,

we need $\lim_{x\to 2} g(x) = g(2)$

$\lim_{x\to 2} g(x)$ exists if $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x)$

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-}\left((x-1)^2 + a\right) = 1+a$$

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+}\left(2^x + ax\right) = 4+2a$$

So $1+a = 4+2a$, that is $a=-3$

We have $\lim_{x\to 2} g(x) = -2 = g(2)$

g is defined as an elementary function on $(-\infty,2)$ and $(2,\infty)$ so it is clearly continuous there.

V1

3. Determine the following limits. *You may use any technique we have seen so far in the course.*

[4pts]

(a) $\lim_{x\to\infty} \frac{x^2+\sqrt{x^6+x^2}}{-x^3+3x}$ $= \lim_{x\to\infty} \frac{x^2+\sqrt{x^6\left(1+\frac{1}{x^4}\right)}}{-x^3+3x}$

$= \lim_{x\to\infty} \frac{x^2+x^3\sqrt{1+\frac{1}{x^4}}}{-x^3+3x} = \lim_{x\to\infty} \frac{\cancel{x^3}\left(\frac{1}{x}+\sqrt{1+\frac{1}{x^4}}\right)}{\cancel{x^3}\left(-1+\frac{3}{x^2}\right)}$

$= \lim_{x\to\infty} \frac{\frac{1}{x}+\sqrt{1+\frac{1}{x^4}}}{-1+\frac{3}{x^2}}$

$= \frac{1}{-1} = \underline{\underline{-1}}$

(b) $\lim_{x\to 3} \frac{(x^2-9)\sin(x-3)}{(x-3)^2}$

$= \lim_{x\to 3} \frac{\cancel{(x-3)}(x+3)\sin(x-3)}{\cancel{(x-3)}(x-3)}$

$= \lim_{x\to 3}(x+3) \cdot \lim_{x\to 3} \frac{\sin(x-3)}{x-3}$

$= 6 \lim_{t\to 0} \frac{\sin t}{t} = 6 \cdot 1 = \underline{\underline{6}}$

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4. Determine each of the following derivatives. *You may use any technique we have seen so far in the course. You do not need to simplify your answer.*

[4pts]

(a) $\dfrac{\mathrm{d}}{\mathrm{d}r}\left(\dfrac{re^r + 3\pi^2}{r^3+1}\right)$

$$= \frac{(re^r + 3\pi^2)'(r^3+1) - (re^r+3\pi^2)(r^3+1)'}{(r^3+1)^2}$$

$$= \frac{(e^r + re^r)(r^3+1) - (re^r+3\pi^2)(3r^2)}{(r^3+1)^2}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\Big(\cos(e^{x^3})\Big)$

$$= \left[-\sin(e^{x^3})\right](e^{x^3})'$$

$$= \left[-\sin(e^{x^3})\right](3x^2 e^{x^3})$$

V1

5. Determine the equations of all lines that are tangent to the curve $f(x) = 2x^2$ and pass through the point $(0, -18)$. *Hint: find the slope of the tangent line to the curve at $x = a$, then find the equation of the tangent line at $x = a$, and then find the values of a that make this line pass through the given point.*

[2pts]

$f'(x) = 4x$

$f'(a) = 4a$

Tangent at $(a, 2a^2)$: $4a = \dfrac{y - 2a^2}{x - a}$

$\Rightarrow y = 4ax - 2a^2$

since the tangent passes through $(0, -18)$:

$-18 = 4a(0) - 2a^2$

$\Rightarrow a^2 = \frac{18}{2} = 9 \Rightarrow \underline{\underline{a = \pm 3}}$

V1

6. A function $y = f(x)$ is defined implicitly using the equation

$$y^5 = e^x + x^2.$$

[2pts] Differentiating implicitly, find an expression for y' in terms of x and y. $\frac{d}{dx}(y^5) = \frac{d}{dx}(e^x + x^2)$

$$5y^4 y' = e^x + 2x$$

$$y' = \frac{e^x + 2x}{5y^4}$$

V1

7. Consider the function

$$f(x) = x^{(e^x)}.$$

[2pts] Find an expression for $f'(x)$ purely in terms of x. *Hint: logarithmic differentiation.*

$$y = x^{e^x}$$

$$\ln y = \ln x^{e^x}$$

$$\ln y = e^x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(e^x \ln x)$$

$$\frac{y'}{y} = e^x \ln(x) + e^x \cdot \frac{1}{x}$$

$$y' = y\left[e^x \ln(x) + \frac{e^x}{x}\right]$$

$$y' = x^{e^x}\left[e^x\left(\ln(x) + \frac{1}{x}\right)\right]$$

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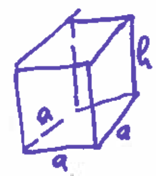
8. A box with a square base is changing its height (h) and the side (a) of its base continuously. When the side (a) measures 2 cm and height (h) measures 9 cm, the side (a) is growing at a rate of 1 cm/min while the height (h) is decreasing at a rate of 2 cm/min. What is

[3pts] the rate of change of the volume at that moment?



side of base : a
height : h

Given $\frac{da}{dt} = 1\,cm/min$ when $(a,h)=(2,9)$

$\frac{dh}{dt} = -2\,cm/min$ when $(a,h)=(2,9)$

We find $\frac{dV}{dt}$ when $(a,h)=(2,9)$

Equation: $V = a^2 h$

$$\frac{dV}{dt} = 2a\frac{da}{dt}h + a^2\frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{(a,h)=(2,9)} = 2(2)(1)(9) + 2^2(-2)$$

$$= 36 - 8 = \underline{\underline{28}}\ cm^3/min$$

Answer: The volume is increasing at a rate of 28 cm^3/min.