1. Let $X = \{(a,b,c) \in \mathbb{R}^3 |\ a + 2b - 3c = 5\}$. Then,

A. X is closed under addition and X is closed under multiplication by scalars.
B. X is not closed under addition and X is also not closed under multiplication by scalars.
C. X is not closed under addition but X is closed under multiplication by scalars.
D. For every vector $\mathbf{v}$ in X, the vector $-\mathbf{v}$ is also in X.
E. $(0,0,0) \notin X$ but X is closed under addition.
F. None of the above statements is true.

Answer: Since $a + 2b - 3c = 5$ is not closed under addition , is not closed under multiplication by scalar, and $(0,0,0) \notin X$. So the answer is B

2. Which of the following are subspaces of $M_{22}(\mathbb{R})$?

$$X_1 = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle|\ a,b,c \in \mathbb{R} \right\}$$

$$X_2 = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle|\ a,b \in \mathbb{Z} \right\} \textit{(Note: } \mathbb{Z} \textit{ is the set of integers.)}$$

$$X_3 = \left\{ \begin{bmatrix} a & 0 \\ 1 & 2a \end{bmatrix} \middle|\ a \in \mathbb{R} \right\}$$

$$X_4 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle|\ a + d = 0; b,c \in \mathbb{R} \right\}$$

A. X_1 and X_2.
B. X_3 and X_4.
C. X_1, X_3, and X_4
D. X_1 and X_3
E. X_1 and X_4
F. X_2 and X_3

Answer: X_2 and X_3 are not closed under multiplication by scalars.
For X_1: $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, so X_1 is a span and it is a subspace.
For X_4: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, so X_4 is a spann, and it is a subspace. Therefore, the answer is E

3. For a fixed choice $\mathbf{v}_1, \ldots, \mathbf{v}_{12}$ of twelve vectors in $\mathbb{R}^{19}$, define

$$X = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{12}\}.$$

Suppose we know that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Then it is always true that:

A. $3 \leq \dim X \leq 12$
B. $3 \leq \dim X < 12$
C. $\dim X = 12$
D. $\dim X = 3$
E. $\dim X > 3$
F. $\dim X < 3$

Answer: The size of any linear independent set in $X \leq \dim \leq$ the size of any spanning set for X so $3 \leq \dim X \leq 12$. therefore, the answer is A

4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

a) Find a basis for the row space of A.

b) Find a basis for the column space of A.

c) Find a basis for $\ker A = \text{Null}(A) = \{x \in \mathbf{R}^4 | Ax = 0\}$.

d) Find the dimension of $\{Ax | x \in \mathbf{R}^4\}$.

Answer: After row reduction:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the $rank(A) = 3$, so a basis for the row space of A is $(1, 1, 0, 1), (0, 1, -1, 1), (0, 1, -1, -1)$. For b. From the echelon form we can see $\{(1, 0, 0)^T, (1, 1, 1)^T, (1, 1, -1)^T\}$ a basis for the column space of A.
For c, we have equations $x_1 = -x_3, x_3 = x_2, x_4 = 0$, the general solution is $\{(x_1, x_2, x_3, x_4) = s(-1, 1, 1, 0)|s \in R\}$, so $(-1, 1, 1, 0)$ is a basis of for $\ker A = \text{Null}(A)$.
For d. Since $col(A) = \{Ax|x \in \mathbf{R}^4\}$, and the dimension of column space of A is 3. So the dimension of $\{Ax|x \in \mathbf{R}^4\}$ is 3.

5. Recall the vector space $\mathbb{P}_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ of polynomial functions of degree at most 2, and define

$$U = \{p \in \mathbb{P}_2 \mid p(2) = 0\}.$$

a) Show that $U = \text{span}\{x - 2, x^2 - 2x\}$. (*Hint: You may wish to use the Factor Theorem: if p is any polynomial and $p(a) = 0$ for some $a \in \mathbb{R}$, then $p(x) = (x - a)q(x)$ for some polynomial q of degree one less than the degree of p.*)

b) Explain why U is a subspace of $\mathbb{P}_2$ without using the subspace test.

c) Find a basis for U.

d) Find $\dim U$.

Solution: (a), If $p \in U$, then by the factor theorem, $(x - 2)$ is a factor of p, here $U = \{p \in \mathbb{P}_2 \mid p(2) = 0\} = \{(x-2)(a+bx) \ |a, b \in R\} = \{a(x-2) + b(x^2 - 2x)|a, b \in R\} = span\{x - 2, x^2 - 2x\}$.

(b) Since $U = span\{x-2, x^2-2x\}$, we know U is a subspace of $\mathbb{P}_2$ (All spans are subspaces)

(c) We claim $\{x-2, x^2-2x\}$ is a basis for U by (a), it remains to show that $\{x-2, x^2-2x\}$ is linear independent. So suppose that $a(x-2) + b(x^2 - 2x) = 0$ for all $x \in R$, rewrite it as: $-2a + (a - 2b)x + bx^2 = 0$, so $a = 0$ and $b = 0$. Therefore $(x - 2), (x^2 - 2x)$ are linear independent. So it is a basis of U.

(d) From (c) $\dim U = 2$.

6. Indicate if each of the following statements is (always) true or is (possibly) false. Put your answers in the boxes indicated and show your work in the spaces below each question.

- If you indicate that the statement is (possibly) false, **you must give an explicit counterexample with numbers!**
- If you indicate that the statement is (always) true, you must give a clear explanation supported with results from class.

a) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in a vector space V and $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.
False: suppose $\mathbf{v}_1 = 0$, $\mathbf{v}_2 \neq 0$, and $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$, but $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent.

ANSWER :

b) If A is a 3×4 matrix, then the columns of A must be linearly dependent.
True : Since A is a 3×4 matrix, then the columns of A is in three dimensional vectors, and we have 4 vectors which is greater than 3, so the columns of A must be linearly dependent.

ANSWER :

6 (continued).

c) If the matrices A and B are invertible, then the matrix $A+B$ is also invertible.
False: Suppose $A = I_2$ and $B = -I_2$. We know both A and B are invertible, however $A+B=0$ is not invertible

ANSWER:

d) Given that $\{x+7, x^2+4x+1, x\}$ is an ordered basis for $\mathbb{P}_2$, the coordinate vector of the polynomial $-2x^2-6x+5$ with respect to this ordered basis is $(1,-2,1)$.

True: We know the coordinate vector with respect to this ordered basis is unique. We know $1 \times (x+7) + (-2) \times (x^2+4x+1) + 1 \times x = -2x^2 - 6x + 5$. so the coordinate vector of the polynomial $-2x^2-6x+5$ with respect to this ordered basis is $(1,-2,1)$.

ANSWER :