MAT 2371, Introduction to probability

Assignment 1 (Chapter 1)

Total = 28 points

Assignment 1: 1.1-6, 1.2-6, 1.2-16, 1.3-12, 1.4-2, 1.5-4.

Problem 1.1-6 (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.6 - 0.4 = 0.7$ *(2 points)*
(b) $P(A \cap B') = P(A) - P(A \cap B) = 0.5 - 0.4 = 0.1$ *(2 points)*
(c) $P(A' \cup B') = P((A \cap B)') = 1 - P(A \cap B) = 1 - 0.4 = 0.6$ *(2 points)*

Problem 1.2-6 *(4 points)*

Suppose D wins, then the last match must be win for D. The previous matches must be combinations of 0 wins for F and 2 wins for D (2!/2!=1), 1 win for F and 2 wins for D (3!/(1!2!)=3, 2 wins for F and 2 wins for D (4!/(2!2!)=6). So the number of ways that the tennis match can end with D winning is 1+3+6=10. Similar there are 10 ways for F to win.

Therefore, there are 20 ways that the tennis match can end.

Problem 1.2-16 *(6 points)*

The number of combinations of $r = 9$ hearts chosen among $n = 52$, without replacement, is

$$\binom{n}{r} = \binom{52}{9}.$$

(a) Suppose that N_1 and r_1 is the total number of white hearts and the number of chosen white hearts, respectively. The number of ways to chose $r_1 = 3$ white hearts chosen among $N_1 = 19$ white hearts is

$$\binom{N_1}{r_1} = \binom{19}{3}.$$

The other $r - r_1 = 6$ hearts are chosen among the other $N - N_1 = 33$ hearts. The possible number ways to chose those hearts is

$$\binom{N - N_1}{r - r_1} = \binom{33}{6}.$$

The probability that exactly 3 of the hearts are white is

$$\frac{\binom{19}{3}\binom{33}{6}}{\binom{52}{9}} = 0.2917.$$

(b) Similarly to (a), the probability that 3 are white, 2 are tan, one is pink, one is yellow, and 2 are green is

$$\frac{\binom{19}{3}\binom{10}{2}\binom{7}{1}\binom{15}{1}\binom{6}{2}}{\binom{52}{9}} = 0.0187.$$

Problem 1.3-12 (a) Let B be the event that we choose blue, and let R be the event that the blue chip is not selected before our selection. Let us consider the collection of chips that were selected before our turn as an unordered sample of chips chosen without replacement.
We choose first: The probability that we select the blue chip is $P(B) = 1/18$. *(1 point)*
We choose fifth: *(2 points)* We have $P(R) = \binom{17}{4}/\binom{18}{4}$, and $P(B|R) = 1/14$. By the product rule, the probability that we select the blue chip is

$$P(B) = P(B|R)P(R) = \frac{1}{14} \cdot \frac{\binom{17}{4}}{\binom{18}{4}} = \frac{1}{14} \cdot \frac{17!\,4!\,14!}{18!\,4!\,13!} = \frac{1}{18}.$$

We choose last: *(2 points)* We have $P(R) = \binom{17}{17}/\binom{18}{17}$, and $P(B|R) = 1$. By the product rule, the probability that we select the blue chip is

$$P(B) = P(B|R)P(R) = 1 \cdot \frac{\binom{17}{17}}{\binom{18}{17}} = 1 \cdot \frac{17!\,1!\,17!}{18!\,0!\,17!} = \frac{1}{18}.$$

Regardless if we choose first, fifth, or last, the probability that we choose the blue chip is 1/18. *(1 point)*

(b) Let B be the event that we choose blue, and let B_j be the event that j blue chips have been selected before our selection, for $j = 0, 1, 2$.

We choose first: The probability that we select the blue chip is $P(B) = 2/18 = 1/9$. *(1 point)*
We choose fifth: *(2 points)* The events B_0, B_1, B_2 are mutually exclusive and exhaustive. Thus, by the total probability rule, we have

$$P(B) = P(B|B_0)P(B_0) + P(B|B_1)P(B_1) + P(B|B_2)P(B_2) = P(B|B_0)P(B_0) + P(B|B_1)P(B_1),$$

since $P(B|B_2) = 0$. We have $P(B_0) = \frac{\binom{16}{4}}{\binom{18}{4}}$, and $P(B|B_0) = 2/14$. Furthermore, we have $P(B_1) = \frac{\binom{2}{1}\binom{16}{3}}{\binom{18}{4}}$, and $P(B|B_1) = 1/14$. The probability that we select the blue chip is

$$P(B) = P(B|B_0)P(B_0) + P(B|B_1)P(B_1) = \frac{2}{14} \cdot \frac{\binom{16}{4}}{\binom{18}{4}} + \frac{1}{14} \cdot \frac{\binom{2}{1}\binom{16}{3}}{\binom{18}{4}} = \frac{1}{9}.$$

We choose last: *(2 points)* In this instance, the events B_1, B_2 are mutually exclusive and exhaustive. Thus, by the total probability rule, we have

$$P(B) = P(B|B_1)P(B_1) + P(B|B_2)P(B_2) = P(B|B_1)P(B_1),$$

since $P(B|B_2) = 0$. We have $P(B_1) = \frac{\binom{2}{1}\binom{16}{16}}{\binom{18}{17}}$, and $P(B|B_1) = 1$. The probability that we select the blue chip is

$$P(B) = P(B|B_1)P(B_1) = 1 \cdot \frac{\binom{2}{1}\binom{16}{16}}{\binom{18}{17}} = \frac{1}{9}.$$

Regardless if we choose first, fifth, or last, the probability that we choose the blue chip is 1/9. *(1 point)*

Alternative solution: Let us consider the collection of chips that were selected before our turn as an **ordered sample** of chips chosen without replacement.

(a) Let B be the event that we choose blue, and let R be the event that the blue chip is not selected before our selection.
We choose first: The probability that we select the blue chip is $P(B) = 1/18$.
We choose fifth: We have $P(R) = {}_{17}\mathrm{P}_4/{}_{18}\mathrm{P}_4$, and $P(B|R) = 1/14$. By the product rule, the probability that we select the blue chip is

$$P(B) = P(B|R)P(R) = \frac{1}{14} \cdot \frac{{}_{17}\mathrm{P}_4}{{}_{18}\mathrm{P}_4} = \frac{1}{14} \cdot \frac{17!\,14!}{18!\,13!} = \frac{1}{18}.$$

We choose last: We have $P(R) = {}_{17}\mathrm{P}_{17}/{}_{18}\mathrm{P}_{17}$, and $P(B|R) = 1$. By the product rule, the probability that we select the blue chip is

$$P(B) = P(B|R)P(R) = 1 \cdot \frac{{}_{17}\mathrm{P}_{17}}{{}_{18}\mathrm{P}_{17}} = 1 \cdot \frac{17!\,1!}{18!\,0!} = \frac{1}{18}.$$

Regardless if we choose first, fifth, or last, the probability that we choose the blue chip is 1/18.

(b) Let B be the event that we choose blue, and let B_j be the event that j blue chips have been selected before our selection, for $j = 0, 1, 2$.

We choose first: The probability that we select the blue chip is $P(B) = 2/18 = 1/9$.
We choose fifth: The events B_0, B_1, B_2 are mutually exclusive and exhaustive. Thus, by the total probability rule, we have

$$P(B) = P(B|B_0)P(B_0) + P(B|B_1)P(B_1) + P(B|B_2)P(B_2) = P(B|B_0)P(B_0) + P(B|B_1)P(B_1),$$

since $P(B|B_2) = 0$. We have $P(B_0) = \frac{{}_{16}\mathrm{P}_4}{{}_{18}\mathrm{P}_4}$, and $P(B|B_0) = 2/14$. Furthermore, to get a permutation of 4 chips with exactly 1 blue, we can select and arrange 3 red chips (number of ways= ${}_{16}\mathrm{P}_3$), select a blue (number of ways= 2), and select the location for the blue chip among the arrangement of red chips (number of ways= 4). Thus, we have $P(B_1) = \frac{{}_{16}\mathrm{P}_3\,(3)(4)}{{}_{18}\mathrm{P}_4}$, and $P(B|B_1) = 1/14$. The probability that we select the blue chip is

$$\begin{aligned} P(B) &= P(B|B_0)P(B_0) + P(B|B_1)P(B_1) = \frac{2}{14}\cdot\frac{{}_{16}\mathrm{P}_4}{{}_{18}\mathrm{P}_4} + \frac{1}{14}\cdot\frac{{}_{16}\mathrm{P}_3\,(2)(4)}{{}_{18}\mathrm{P}_4} \\ &= \frac{2}{14}\cdot\frac{16!\,14!}{12!\,18!} + \frac{1}{14}\cdot\frac{16!\,14!\,(2)(4)}{13!\,18!} = \frac{1}{9}. \end{aligned}$$

We choose last: In this instance, the events B_1, B_2 are mutually exclusive and exhaustive. Thus, by the total probability rule, we have

$$P(B) = P(B|B_1)P(B_1) + P(B|B_2)P(B_2) = P(B|B_1)P(B_1),$$

since $P(B|B_2) = 0$. To get a permutation of 17 chips with exactly 1 blue, we can select and arrange 16 red chips (number of ways= ${}_{16}\mathrm{P}_{16}$), select a blue (number of ways= 2), and select the location for the blue chip among the arrangement of red chips (number of ways= 17). Thus, we have $P(B_1) = \frac{{}_{17}\mathrm{P}_{17}\,(2)(17)}{{}_{18}\mathrm{P}_{17}}$, and $P(B|B_1) = 1$. The probability that we select the blue chip is

$$P(B) = P(B|B_1)P(B_1) = \frac{{}_{16}\mathrm{P}_{16}\,(2)(17)}{{}_{18}\mathrm{P}_{17}} = \frac{1}{9}.$$

Regardless if we choose first, fifth, or last, the probability that we choose the blue chip is 1/9.

Problem 1.4-2 *(6 points)*
(a) Since A and B are independent, then

$$P(A\cap B) = P(A)P(B) = (0.2)(0.7) = 0.14.$$

(b) Thus

$$P(A\cup B) = P(A) + P(B) - P(A\cap B) = 0.2 + 0.7 - 0.14 = 0.76.$$

(c) Since $A'\cup B' = (A\cap B)'$, thus

$$P(A'\cup B') = 1 - P(A\cap B) = 1 - 0.14 = 0.86.$$

Problem 1.5-4 *(4 points)*
Let G, G_2, G_3, G_4 be the event that the selected driver is in the age group 16-25, 26-50, 51-65, 66-90, respectively. Let A be the event that the selected driver had an accident. By the total probability rule, the probability that the selected driver had an accident is

$$\begin{aligned} P(A) &= P(A|G_1)P(G_1) + P(A|G_2)P(G_2) + P(A|G_3)P(G_3) + P(A|G_4)P(G_4) \\ &= (0.05)(0.10) + (0.02)(0.55) + (0.03)(0.20) + (0.04)(0.15) = 0.028. \end{aligned}$$

By Bayes' rule, the probability that the selected driver is in age group 16-25 given that they had an accident is

$$P(G_1|A) = \frac{P(A|G_1)P(G_1)}{P(A)} = \frac{(0.05)(0.10)}{0.028} = 0.1786.$$